Langevin models of Turbulence

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1 Introduction

In a turbulent flow, the number of degree of freedom N can be gigantic, scaling as the 9/4 power of the Reynolds number. In the atmosphere, this number may reach $N \sim 10^{16}$, devastating our hope to implement all scales of the climate system onto a computer. This juggling with numbers illustrates the well known challenge posed by turbulent flows: is there a way to simulate, or describe a turbulent flow, without taking into account all degrees of freedom? A similar question has been asked in the past by founders of statistical mechanics. Specifically, it has been the kind of challenge met by Boltzman and co-workers to describe the behavior of a gas made of billions of particles. Of course, in the case of turbulence, an additional difficulty arises because a turbulent flow is necessarily driven out of equilibrium by the energy input mechanisms. Therefore, none of the well-known recipes of classical statistical mechanics apply. Yet, we may learn something from our glorious ancestors by closer inspection of their protocol: in a gas, the number of particles is so huge that it is just hopeless trying to follow each of them individually. Whatever our power of measurements, there will remain individual particles which we will be unable to follow. Instead of starting an endless race towards finer and finer measurements, aimed at decreasing their corresponding number, why not accept this inherent ignorance, and replace it by something mimicking its action, and which will be easy to handle? This is precisely the reasoning followed by Langevin, upon modeling the Brownian motion by a simple Gaussian white noise. Such simple rule achieved many successes. Can it be simply translated to turbulent flows? This possibility is discussed in the present short review. Additional point of view about this may be found in the contribution by Friedrich in these proceedings.

2 Langevin models of turbulence

2.1 Framework

Consider a turbulent flow, with velocity field $v_i(x,t)$, and introduce an (arbitrary) filtering procedure so as to separate it into a large-scale field $U_i = \overline{v_i}$ and a small-scale component $u_i = v_i - U_i$. Such small-scale motion varies over time scale t, while large scale vary over time scale T. In any reasonable turbulent flow, the ratio of the typical time scale of the two components varies like a power of the scale ratio, as $t/T \sim (l/L)(U/u) \sim (l/L)^{2/3}$. Therefore, small scales vary much more rapidly than large scale. From the point of view of the largest scales, the small scales may then be regarded as a noise. Hence the idea to simply replace them by an a priori chosen noise, with well-defined properties. One classical way is though a generalized Langevin equation:

$$\dot{u_i} = A_{ij} u_j + \xi_i, \tag{1}$$

where A is a generalized friction operator, and ξ is a noise. In the sequel, we explore various models characterized by different value of A and ξ.

2.2 Obukov model

The simplest model one can imagine is to take $A = 0$ and ξ as a Gaussian white noise, isotropic and homogeneous in space, with short time correlation:

$$< \xi_i(x,t)\xi_j(x',t) >= 2\Delta\delta_{ij}\delta(t-t'). \tag{2}$$

This model has been first introduced in 1959 by Obukhov. It leads to a number of interesting properties.

Richardson law and Kolmogorov spectrum

Consider for example a cloud of passive scalar particles, embedded in such a flow. After a time t, this cloud of particles will have evolved into a situation where its velocity distribution obeys a Gaussian statistics, with variance scaling like square root of time: $\delta u = \sqrt{< u^2 > - < u >^2} \sim t^{1/2}$. In parallel, the cloud of particles experienced a spread by a factor $r \sim \sqrt{< x^2 > - < x >^2} \sim t^{3/2}$. This last law is nothing but the famous Richardson law, an empirical law describing the dispersion of passive tracers in the atmosphere. Moreover, we may combine the two simple relation to obtain that $\delta u \sim r^{1/3}$, implying a velocity spectrum $E(k) \sim k^{-5/3}$, i.e. the Kolmogorov spectrum. We see that with virtually no effort, Obukhov model reproduces the two more robust experimental results obtained so far in turbulence!

Limitations

Richardson law and Kolmogorov spectra are representative of velocities which do not differ from the mean by a large amount. The actual range of validity of the Obukhov model arises when considering higher moments, involving rarer, but more violent events. Since velocities in this model are Gaussian, their moments obey a simple scaling relation $:< u^{2n} >\sim< u^2 >^n$, at variance with the intermittency observed in real turbulent flows. This simple hierarchy law disappears as soon as one allows for spatial or temporal correlation, as recently proved in the Kraichnan model of turbulence.

2.3 Kraichnan model

The Obukhov model is frictionless in essence. The Kraichnan model can be viewed as the opposite limit, with a very large friction $A_{ij} = -\gamma\delta_{ij}$, $\gamma \ll 1$, and a noise with spatial correlation

$$< \xi_i(x,t)\xi_j(x',t) >= 2\Delta_{ij}(x,x')\delta(t-t'). \tag{3}$$

Due to the large friction, the inertial term in the Langevin equation becomes negligible and the velocity adiabatically adjusts to the noise as: $u_i \sim \gamma\xi_i$. The Kraichnan model is thus made of small-scale delta-correlated Gaussian white noise, with spatial correlation.

Intermittency and conservation laws

Contrarily to Obukhov model, Kraichnan model leads to intermittency for the high order moments. The physical reasons have been recently reviewed in [1]. They are rooted in the spatial correlation, which induce a memory effect onto lagragian trajectories, and lead to the apparition of conservation laws within sets of lagrangian particles. Since the moment of order $2n$ is associated with conservation laws of sets of $2n$ particles, and since conservation laws of sets of particles of different sizes are not simply related, this induces a breaking of the hierarchical structure of the moments.

Turbulent transport

Another less well known property of Kraichnan model concerns turbulent transport. Suppose we focus on the evolution of the vorticity in such a model. In classical turbulence, the vorticity obeys the equation

$$\partial_t\Omega_i = -v_k\partial_k\Omega_i + \Omega_k\partial_k v_i + \nu\partial_k\partial_k\Omega_i, \tag{4}$$

where ν is the molecular viscosity, and v is the sum of the large scale component U and the (small-scale) noise. Because of the presence of noise, eq.

(4) admits stochastic solution, whose dynamic can be fully specified by the probability distribution function. Ignoring the viscosity and using standard techniques [2], one can derive the evolution equation for $P(\mathbf{\Omega}, x, t)$, the probability of having the field $\mathbf{\Omega}$ at point x and time t:

$$\begin{aligned} \partial_t P = & -U_k \partial_k P - (\partial_k U_i)\partial_{\Omega_i}[\Omega_k P] + \partial_k[\beta_{kl}\partial_l P] \\ & + 2\partial_{\Omega_i}[\Omega_k \alpha_{lik}\partial_l P] \\ & + \mu_{ijkl}\partial_{\Omega_i}[\Omega_j \partial_{\Omega_k}(\Omega_l P)] \end{aligned} \tag{5}$$

For simplicity, we assumed homogeneity of the fluctuations and we introduced the following turbulent tensors:

$$\begin{aligned} \beta_{kl} &= \langle u_k u_l \rangle \\ \alpha_{ijk} &= \langle u_i \partial_k u_j \rangle \\ \mu_{ijkl} &= \langle \partial_j u^i \partial_l u^k \rangle \end{aligned} \tag{6}$$

Due to incompressibility, the following relations hold: $\alpha_{kii} = \mu_{iikl} = \mu_{ijkk} = 0$.

To illuminate the signification of this complicated equation, let us consider the first moment of eq. (5), obtained by multiplication with Ω_i and integration:

$$\begin{aligned} \partial_t \langle \Omega_i \rangle = & -U_k \partial_k \langle \Omega^i \rangle + (\partial_k U_i)\langle \Omega_k \rangle - 2\alpha_{kil}\partial_k \langle \Omega_l \rangle \\ & + \beta_{kl}\partial_k \partial_l \langle \Omega_i \rangle. \end{aligned} \tag{7}$$

In addition to the standard vorticity advection and stretching by the large scale, one recognize two additional effect: one proportional to α, resulting in large-scale vorticity generation through the AKA instability [3]; one proportional to β, akin to a turbulent viscosity. Within the Kraichnan model, one therefore naturally recovers the well-known formulation of turbulent transport, without resorting to scale separation [4]. In this very simple model, where the viscosity has been ignored, one can show that the tensor β is always positive: the turbulent viscosity always enhances turbulent transport. In actual viscid flows, the turbulent viscosity tensor is actually fourth order, and can be negative [4].

Limitation

This digression about turbulent transport shows that the way we prescribe velocity correlation in Kraichnan model somehow determines the turbulent transport properties of the flow. It is a kind of adjustable parameter. In that respect, it would be nice to devise a model devoid of this freedom of choice, by ensuring for example that the turbulent transport somehow adjusts itself to the way energy is injected and dissipated, as in real turbulence. In the sequel, we present a model where the noise is dynamically computed at each time scale, thereby removing the arbitrariness of the Langevin model.

2.4 Stochastic RDT model

Description

Our method is based on the observation that small scales are mostly slaved to large scale via linear processes akin to rapid distortion. This observation is substantiated by various numerical simulations and is linked with the prominence of non-local interactions at small scale [5]. Specifically, let us decompose our small-scale velocity field into wave packets, via a localized Fourier transform:

$$\hat{u_i}(x,k) = \int h(x-x')e^{ik(x-x')}u_i(x')dx',$$

where h is a filtering function, which rapidly decays at infinity. Using incompressibility and non-locality of interaction, one can derive the following equation of motion for the wave-packet [5, 6]:

$$\begin{aligned}
\dot{x_i} &= U_i, \\
\dot{k_i} &= -k_j \partial_j U_i, \\
\dot{\hat{u_i}} &= -\nu_t k^2 \hat{u_i} + \hat{u_j}\partial_j \left(2\frac{k_i k_m}{k^2} U_m - U_m \right) + \hat{\xi_i}.
\end{aligned} \tag{8}$$

Here ν_t is a turbulent viscosity describing the local interactions between small-scales, and ξ is a forcing stemming from the energy cascade. Its expression only involves large-scale non-linearities (in fact the aliasing) via $\xi_i = \partial_j \left(U_j U_i - \overline{U_i U_j} \right)$. By eq. (8) the wave-packet is transported by the large-scale flow, its local wavenumber is distorted by the large-scale velocity gradients, and its amplitude is modified through the action of local and non-local interactions. The equation describing its amplitude evolution is a generalized Langevin equation, with friction generated by turbulent viscosity and with both multiplicative and additive noise stemming from interaction with large scale. Because these two noises are of same origin, they are correlated. One can show that this correlation is responsible for a skewness in the probability distribution of the small scale [5]. Note also that in some sense, our Langevin model can be viewed as a generalization of Rapid Distortion Theory equation, with inclusion of turbulent viscosity and stochastic forcing. No wonder, interesting analytical properties will be available in precisely the same case where Rapid Distortion Theory is the most useful, namely rotating, or stratified shear flows (see below).

Equation (8) shows that our model is specified by the knowledge of ξ and U_i. The latter can be shown by mere filtering to obey the equation:

$$\partial_t U_i + U_j \partial_j U_i = -\partial_i \bar{p} + \nu \Delta U_i + \bar{f_i}, \tag{9}$$

where f_i describes the backreaction of small scales onto large scales, and is obtained through summation over wavepackets:

$$f_i(x) = \int dk \partial_j \left(\overline{U_i(x)\hat{u_j}(x,k)} + (\overline{\hat{u_i}(x,k)U_j(x)} + (\overline{\hat{u_i}(x,k)\hat{u_j}(x,-k)} \right) .$$

The set of eqs. (8) and (9) is a strongly non-linear system of coupled equations, which defines our turbulence langevin model. In this method, the noises can be dynamically computed at each time step by integration of the large -scale equation. In the sequel, we present two application of this: one in which the system is simplified by *prescribing* one of the forcing (namely ξ). This allows the computation of general scaling laws for turbulent transport in various systems. One in which ξ is numerically computed using the large scale equation. This allows for fast numerical simulations.

Turbulent transport

We consider here a simplified version of our Langevin model where the function ξ is not computed, but prescribed as a Gaussian delta-correlated white noise. The advantage of this simplification is that it allows for analytical computation in special situation, where the geometry of the system is so simple that it allows for explicit solution of the homogeneous (unforced) small-scale equations. In some sense, our model with prescribed Gaussian model for ξ can be viewed as a generalization of Kraichnan model of turbulence. One can then expect this model to provide "reasonable shape" for turbulent transport (see Section 2.3), with "free" parameter induced by the prescribed correlation function for ξ. Working out the details, we found out that indeed, our model is able to provide the scaling of the turbulent transport, as a function of control parameters, up to a numerical prefactor, controlled by the intensity of the correlation of the forcing ξ (the "free parameter").

Heat transport in convection

When a horizontal layer of fluid is heated from below, a heat exchange from the top to the bottom occurs. The transport of heat depends on the interplay between the thermal, viscous and integral scales of turbulence, and thus, on both the Prandtl number and the Reynolds numbers. Our model can be used to predict both the structure and the scaling laws in thermal convection [7]. In the boundary layer, the velocity profile is logarithmic and the temperature decays like the inverse of the distance to the wall. This has important impact onto the heat transport. At low Reynolds numbers, when most of the dissipation comes from the mean flow, we recover power classical scaling regimes of the Nusselt versus Rayleigh number, with exponent 1/3 or 1/4. At larger Reynolds number, velocity and temperature fluctuations become non-negligible in the dissipation. In these regimes, there is no exact power law dependence the Nusselt versus Rayleigh or Prandtl. Instead, we obtain logarithmic corrections to the classical soft (exponent 1/3) or ultra-hard (exponent 1/2) regimes, in a way consistent with the most accurate experimental measurements available nowadays. Example is given in the figure 1, showing the comparison between the data of the Castaing group in Helium, versus the theoretical predictions (lines).

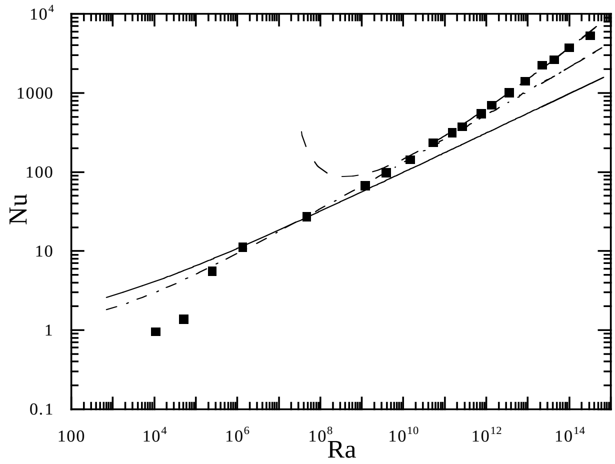

Fig. 1. Illustration of the three scaling regimes found in convection in Helium for Nusselt vs Rayleigh. The symbols are experimental measurements by [8]. The lines are theoretical prediction by [7] using an analytical model of turbulent convection. "Soft" turbulence regime(mean flow dominated): power law $Nu \sim Ra^{1/4}$ (full line); "Hard" turbulence regime: (velocity fluctuation dominated) $Nu \sim Ra^{1/3}/(\ln(Ra))^{2/3}$ (dotted line); "Ultra-hard" turbulent regime: (temperature fluctuations dominated) $Nu \sim Ra^{1/2}/(\ln(Ra))^{3/2}$ (dashed line)

The theory has also been extended to describe turbulent thermal convection at large Prandtl number [9]. Two regimes arise, depending on the Reynolds number Re. At low Reynolds number, $NuPr^{-1/2}$ and Re are a function of Ra $Pr^{-3/2}$. At large Reynolds number $NuPr^{1/3}$ and $RePr$ are function only of $RaPr^{2/3}$ (within logarithmic corrections). In practice, since Nu is always close to $Ra^{1/3}$, this corresponds to a much weaker dependence of the heat transfer in the Prandtl number at low Reynolds number than at large Reynolds number. This difference may solve an existing controversy between measurements in SF6 (large Re) and in alcohol/water (lower Re). These regimes may be linked with a possible global bifurcation in the turbulent mean flow. A scaling theory can be used to describe these two regimes through a single universal function. This function presents a bimodal character for intermediate range of Reynolds number. This bimodality can be explained in term of two dissipation regimes, one in which fluctuation dominate, and one in which mean flow dominates. Altogether, our results provide a six parameters fit of the curve $Nu(Ra, Pr)$ which may be used to describe all measurements at $Pr > 0.7$.

Momentum transport in rotating shear flow

At sufficiently large Reynolds number, the fluid between co-rotating coaxial cylinders becomes turbulent, and a significant momentum transport occurs between the two cylinders. In the case with rotating inner cylinder and resting outer one (the so-called Taylor-Couette flow), detailed measurements show that the torque applied at cylinders by the turbulent flow is a function of the Reynolds number R. Within the Langevin model, one can work out an analogy between the problem of momentum transport and heat transport in turbulent convection, to compute the torque in Taylor-Couette configuration, as a function of the Reynolds number [10]. At low Reynolds numbers, when most of the dissipation comes from the mean flow, we predict that the non-dimensional torque $G = T/\rho\nu^2 L$, where L is the cylinder length, scales with Reynolds number R and ratio of inner cylinder to outer cylinder radius $\eta = r_i/r_o$, $G = 1.46\eta^{3/2}(1-\eta)^{-7/4}R^{3/2}$. At larger Reynolds number, velocity fluctuations become non-negligible in the dissipation. In these regimes, there is no exact power law dependence the torque versus Reynolds. Instead, we obtain logarithmic corrections to the classical ultra-hard (exponent 2) regimes:

$$G = 0.50\frac{\eta^2}{(1-\eta)^{3/2}}\frac{R^2}{\ln[\eta^2(1-\eta)R^2/10^4]^{3/2}}.$$

These predictions are found to be in excellent agreement with available experimental data (see figure 2).

Fast numerical simulations

We consider now the case where ξ is not prescribed, but dynamically computed using the large scale equation. In that case, there is no free parameter in the problem, except for the magnitude of the turbulent viscosity. By comparison with direct numerical simulation, we found however [6] that in isotropic case, the magnitude of this turbulent viscosity can be prescribed using the DSTA model of Kraichnan

$$\nu_t(k) = C_t\left(\frac{k}{k_i}\right)^{-4/3}\sqrt{\frac{E(k_i)}{k_i}}, \qquad (10)$$

where E is the energy spectrum, k_i is a wavenumber in the inertial range and C_t is a constant depending on a parameter β characterizing the degree of non-locality of the interaction. For the contribution at k of all modes with wavenumber greater than βk, it yields:

$$C_t(\beta) = \sqrt{7/60}\,\beta^{-2/3} = 0.3416\,\beta^{-2/3}, \qquad (11)$$

with β depending on the ratio of the largest wave-number of the (resolved) simulation onto the cut-of wavenumber as $\beta = k_{max}/k_c$.

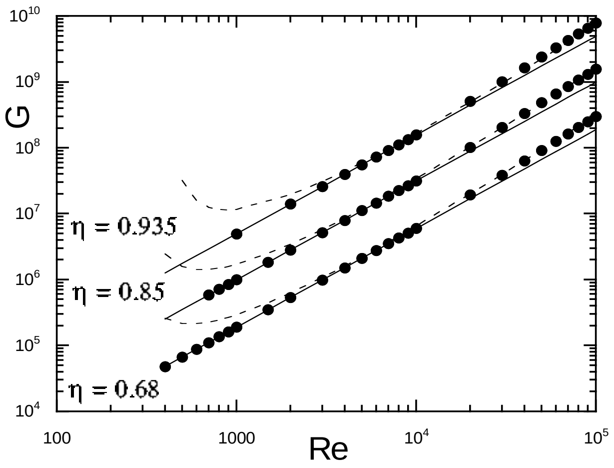

Fig. 2. Torque vs Reynolds in Taylor-Couette experiments for different gap widths $\eta = 0.68$, $\eta = 0.85$ and $\eta = 0.935$. The symbols are the data of [11]. The lines are the theoretical formula obtained in the soft and ultra-hard turbulence regimes and computed using the analogy with convection. Soft turbulence (full line); ultra-hard turbulence (dotted line). There is no adjustable parameter in this comparison, all the constants being fixed either by the analogy with convection, or by the comparison with the data of [12].

With this prescription, we may then see our Langevin model as a parameter-free model of turbulence. Its formulation is rather complex, but its advantage lies in the possibility to use a semi-lagrangian scheme of integration for the small scale, thereby allowing for very large time steps. As a result, we obtain a fast numerical simulation, with all scales being resolved, but with an integration time smaller by a factor 10 to 1000 with respect to traditional DNS [13]! An example is provided on Figure 3 in the case of 2D turbulence.

3 Towards a LES langevin model?

In this short review, we hope to have convinced you of the interest of Langevin models of turbulence. However, we did not yet fully achieve the goal we fixed in the introduction: our model still retains infinitely many degrees of freedom, symbolized by unrestrained number of wave-packets we use. In some situations, this feature is more than desirable: a peasant working on his crop is seldom interested by the weather forecast at the level of his country, and would like to know the hail forecast at the level of his field! However, in most applications, one does not need such a wealth of details, and one would prefer a Langevin model with very few degrees of freedom. Such a "large eddy

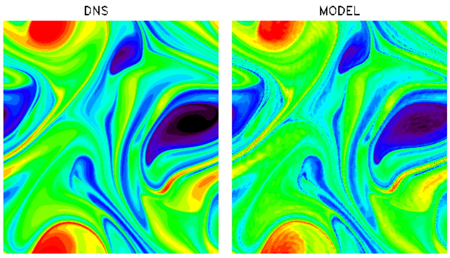

Fig. 3. Fast 2D numerical simulation. Left panel: vertical vorticity $\omega_z = \nabla \times \mathbf{u} \cdot e_z$ computed using standrad spectral method. This simulation required 3 days to be completed on our workstation. Right panel: same field, computed using our Langevin method. This was obtained in only 30 minutes, on the same work station.

Langevin" model remains to be built. We are currently working in that direction.

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