How to use the idea of Leray singularity to account precisely for the observed and measured intermittency in turbulent flows

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Outilne.1

Turbulence theory challenges our understanding of what is a (smooth) function in the sense of Leibnitz and how to deal with the lack of smoothness of turbulent velocity fields that seems to be observed. Two different ways of dealing with this question:
- assume that the velocity field is continuous but non differentiable function of position like the Weierstrass counter-example.
- Follow Leray's idea (1934) and look at possible singularities of the Euler and/or Navier-Stokes equations that are local in space and time.

Outline.2

- What are self-similar solutions of Euler equations? A schema to derive them explicitly.
- How can singularities be observed (indirectly) in time records of velocity fluctuations at a single point in a turbulent flow (Modane wind tunnel)
- How to put the singular solutions in a coherent schema for understanding turbulent flows: the example of structure functions.

Weierstrass counter example (1872) and turbulent velocity field.1

Weierstrass (1872) example of a period-2 *continuous* function of t but *non differentiable* almost everywhere (original notations)

$$f(t) = \Sigma_\nu b^\nu \cos(a^\nu \pi t),$$

ν set of positive integers.
$f(t)$ is continuous (non trivial) and non differentiable (easier) almost everywhere if a positive integer and

$$ab > 1 + 3\pi/2.$$

Weierstrass counter example (1872) and turbulent velocity field.2

Application to turbulence: If K41 scaling holds true $\delta u(x) \sim (\delta x)^{1/3}$. Therefore $\delta u(x)/\delta x \sim (\delta x)^{-2/3}$ and $u(x)$ is continuous and non differentiable as in Weierstrass counter-example. This is inconsistent with the idea of a nowhere differentiable solution of NS or Euler equations: $\partial_t \delta u(x,t) \sim \delta u(x) \nabla \delta u(x) \sim ((\delta x)^{1/3})^2/(\delta x) \sim (\delta x)^{-1/3}$, then $\delta u(x, t+\Delta) \sim \Delta(\delta x)^{-1/3}$, Δ small. This is inconsistent with the starting point (K41 scaling)

The same reasoning imposes a smooth velocity field almost everywhere:

If $\delta u(x) \sim (\delta x)^{\eta} \geq \eta \leq 1$ and if one imposes $\partial_t \delta u(x,t) \sim \delta u(x) \nabla \delta u(x)$ one finds $\eta = 1$ (differentiable velocity field).

The fundamental question of turbulence theory

Do flows of incompressible fluids in 3D at large or infinite Reynolds number (namely at small or zero viscosity) display finite time singularities localized in space, or are they continuous and non differentiable almost everywhere as Weierstrass counter example?
Single-point records of velocity fluctuations display correlations between large velocities and large accelerations in full agreement with scaling laws derived from Leray-like equations (1934) for self-similar singular solutions to the fluid equations (Euler-Leray equations). Conversely, those experimental velocity - acceleration correlations strongly contradict Kolmogorov scaling laws.
Moreover the so-called structure functions for the acceleration display a remarkable transition at increasing power of the fluctuation, well explained by supposing the flow made of individual Leray-like singular events (almost) independent of each other.
No cut-off of singularities at small scales by viscosity in the usual meaning.

Leray's singularities

The Euler-Leray equations for self-similar singular solutions of an inviscid incompressible fluid are derived from the Euler equations. The similarity exponents take into account either Kelvin's theorem of conservation of circulation or energy conservation (if energy is finite)

1) What are Euler-Leray equations? + strategy for an explicit (analytical) solution.

2) Amazing agreement between predictions of Euler-Leray with intermittency seen in velocity fluctuations in Modane wind tunnel.

Dissipation by localized singularities in other settings: shock waves in compressible fluids, white caps of gravity waves, NLS focusing equation (next talk by Christophe Josserand).

Challenge (+ work in progress): put localized (space and time) dissipation in a coherent statistical framework. Same transition in structure functions as exponent increases seen both in turbulent data and in numerical studies of focusing NLS.

Derivation of Leray's equations.1

In 1934 Jean Leray ("Essai sur le mouvement d'un fluide visqueux emplissant l'espace", Acta Math. **63** (1934) p. 193 - 248) published a paper on the equations for an incompressible fluid in 3D. He introduced many ideas, among them the notion of weak solution and also what problem should be solved to show the existence (or not) of a solution singular at a point after a finite time with smooth initial data.

Leray assumed a solution of 3D Navier-Stokes (NS) blowing-up in finite time at a point, following self-similar evolution for smooth and uniformly bounded initial velocity field. Unknown yet if this solution exists, either for Euler and/or NS.

Derivation of Leray's equations.2

Euler equations (inviscid, incompressible, 3D):

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p,$$

and

$$\nabla \cdot \mathbf{u} = 0,$$

Leray looked (with viscosity added, Navier-Stokes equations) to self-similar solutions of the type:

$$\mathbf{u}(\mathbf{r}, t) = (t^* - t)^{-\alpha} \mathbf{U}(\mathbf{r}(t^* - t)^{-\beta}),$$

where t^* is the time of the singularity (set to zero), where α and β are positive exponents to be found and where $\mathbf{U}(.)$ is a numerical function to be derived by solving Euler or NS equations.
That such a velocity field is a solution of Euler or NS equations implies $1 = \alpha + \beta$. The conservation of circulation in Euler equations implies $0 = \alpha - \beta$, and $\alpha = \beta = 1/2$. If one imposes instead that a finite energy in the collapsing domain is conserved, one must satisfy the constraint $-2\alpha + 3\beta = 0$, which yields $\alpha = 3/5$ and $\beta = 2/5$, the Sedov-Taylor exponents.

Derivation of Leray's equations.3

No set of singularity exponents can satisfy both constraints of energy conservation and of constant circulation on convected closed curves. $\alpha = \beta = 1/2$ if there are smooth curves invariant under Leray stretching. Same exponents found by Leray for NS. Otherwise one has to take the Sedov-Taylor scaling, assuming that
1) the collapsing solution has finite (or logarithmically diverging as here) energy,
2) no closed curve is carried inside the singular domain while keeping finite length and remaining smooth.
Important remark: unknown if there is a single set of values of exponents. A possible interpretation of experimental data is that a spectrum of values of exponents, including with a negative α between 0 and (-1). The velocity is continuous but not the acceleration if α is in this range. However this can be also understood as an effect of viscosity: otherwise the conservation of energy imposes the Sedov-Taylor scaling.

Derivation of Leray's equations.4

Introduce boldface letters such that $\mathbf{R} = \mathbf{r}(-t)^{-\beta}$. The Euler equations become the Euler-Leray equations for $\mathbf{U}(\mathbf{R})$:

$$-(\alpha\ \mathbf{U} + \beta\ \mathbf{R} \cdot \nabla\mathbf{U}) + \mathbf{U} \cdot \nabla\mathbf{U} = -\nabla P,$$

and

$$\nabla \cdot \mathbf{U} = 0$$

A general time dependence can be kept besides the one due to the rescaling of the velocity and distances by defining as new time variable $\tau = -\ln(t^* - t)$. This maps the dynamical equation into

$$\frac{\partial\mathbf{U}}{\partial\tau} - (\alpha\mathbf{U} + \beta\mathbf{R} \cdot \nabla\mathbf{U}) + \mathbf{U} \cdot \nabla\mathbf{U} = -\nabla P,$$

$$\nabla \cdot \mathbf{U} = 0$$

Equivalent to the original Euler equations.

Explicit solution of Euler-Leray equations: an outline.1

Euler-Leray equations in axisymmetric geometry with swirl and possibly periodic dependence on τ (work in progress + Pomeau-Le Berre in arXiv):

1) Start from a localized solution of steady localized Euler equation by solving Hicks equation(1898). Because this has finite energy one takes Sedov-Taylor exponents.

2) Because steady Euler equations are invariant under arbitrary dilations of amplitude or argument (being homogeneous of order 2 and invariant under dilation of coordinates) one can assume that the solution of Hicks equation has very large amplitude.

3) This makes the (linear) streaming term added by Leray arbitrarily small compared to the leading order term which is quadratic.

4) Solving Euler-Leray by perturbation one meets two solvability conditions because of the two dilation symmetries of the steady Euler equations. They can be satisfied either by adding two small oscillations with arbitrary amplitudes or by tuning free coefficients of the unperturbed solution of Hicks equation.

Explicit solution of Euler-Leray equations: an outline.2

A few points on this solution of Euler-Leray equations:
- The expansion of the solution at higher orders is formally well defined because at each order one can add to the base solution a contribution belonging to the 2D kernel of the linearized problem.
- Solutions of Euler-Leray show a divergence of vorticity, because vorticity scales like $1/(t_* - t)$, so there is (well hidden) vortex stretching in this solution.
- Hicks equation reads:

$$(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r})\Psi - r^2 \frac{\mathrm{d}H}{\mathrm{d}\Psi} + B\frac{\mathrm{d}B}{\mathrm{d}\Psi} = 0,$$

with $H(.)$ and $B(.)$ arbitrary functions of the stream function Ψ for the flow in the (r, z) half plane (without boundary conditions). One can choose (as did Hicks) H and B to make Hicks equation linear and decompose in Fourier transform along the $z-$ direction. This leaves a free weight function of the wavenumber along z.
- Historical point: Hicks tried to find a kind of Schrodinger-like equation for Kelvin's vortex model of atoms.

Is it possible to "observe" Euler-Leray singularities?.1

Our motivation for working on Euler-Leray singularities is their possible connection with the phenomenon of intermittency in high Reynolds number flows. This raises several questions:
1. What is specific to Leray singularities compared to other schema for intermittency?
2. What would be specific of an Euler-Leray singularity in time records of single point velocity in a large Reynolds number flow ?
3. What are precisely the consequences of the occurrence of Leray-like singularities on the statistics of a turbulent flow?

Is it possible to "observe" Euler-Leray singularities?.2

Point 1 : If intermittency is caused by Leray-like singularities, they should yield strong *positive* correlation between singularities of the velocity and of the acceleration. This is what is observed.
Compared to scaling prediction derived from Kolmogorov-like exponents this correlation is a strong indication of the occurrence of singularities near large fluctuations. Moreover Kolmogorov theory extended to dissipative scales excludes exponents of the singularity of the velocity fluctuations vs distance which is less than 1/3: otherwise dissipation is divergent everywhere in space, clearly impossible.
The only way-out is to have dissipative events at random points in space *and* time instead of being always spread everywhere (as singularities of the derivative in the counter example of Weierstrass).

Euler-Leray singularities and intermittency.1

Kolmogorov K41 theory is based upon the idea that turbulent fluctuations at very large Reynolds number (where the effect of viscosity is formally small) depend on the power dissipated in the turbulent flow per unit mass, ϵ.
K41 is successful for predicting the spectrum of velocity fluctuations (Kolmogorov-Obukhov spectrum $k^{-5/3}$) but is contradicted by intermittency. Because of it the fluctuations fail to satisfy the relationship predicted by Kolmogorov between the velocity fluctuation and the distance between two points of measurement. Using the scaling law with ϵ, one finds $< (u(r_0 + r, t) - u(r_0, t))^3 > \sim (\epsilon r)$ when the distance r is in the (wide) range between the largest scales and the length scale short enough to make the viscosity relevant. If applied to arbitrary power n this predicts that, as r gets smaller and smaller, the amplitude of the velocity fluctuation decreases, not what is observed. K41 scaling fails badly as soon as $n > 4$.
Statistical theory based on random occurrence of Leray-like singularities (see later).

Euler-Leray singularities and intermittency.2

We have very long and high quality records of velocity fluctuations in the high-speed wind tunnel of Modane in the French Alps, obtained by hot-wire anemometry (Yves Gagne et al. 1998), and all sorts of correlations can be studied.

Suppose the observed large bursts of velocity are due to Euler-Leray singularities. It means that $u(r,t)$ scales like $(-t)^{-\alpha}$ as t tends to zero (0 taken arbitrarily as the instant of the singularity). The acceleration γ (time derivative of Eulerian u) scales like $(-t)^{-(1+\alpha)}$ as t tends to zero. Therefore near the singularity both the velocity and the acceleration diverge, this latter the most strongly and in this large burst u^3 is of order γ if conservation of circulation is taken:

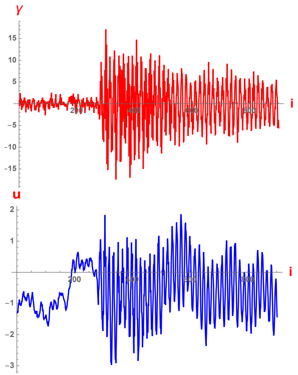
$$u^3 \sim \Gamma\gamma$$

The multiplicative constant is of the order of a "typical" value of the circulation. With the Sedov-Taylor exponents, on has instead:

$$u^8 \sim E\gamma^3$$

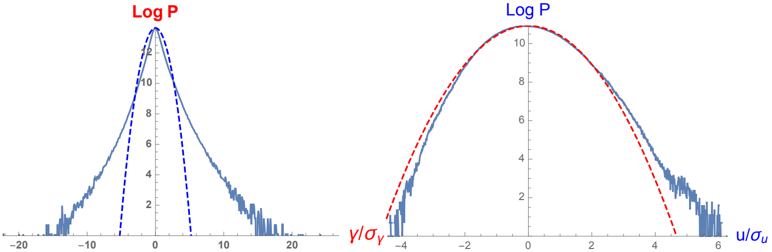
where E is the energy inside the collapsing domain.

burst from Modane 2014; $\gamma(t)$ (red); $u(t)$ (blue)



$\gamma/g = 56000$; (Maximum ratio $\gamma/g = 10^6$ for Modane-2014 ; and $\gamma/g = 6000$ for Modane-1998) g acceleration of gravity.

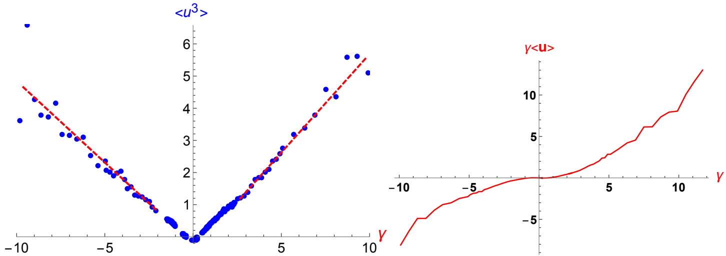
Gaussian Statistics?



Non Gaussian acceleration Slightly non Gaussian velocity

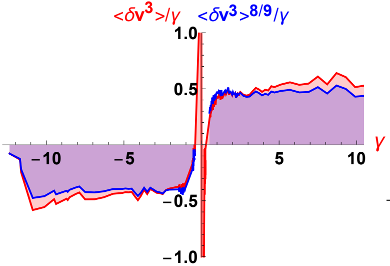
Common situation due to the presence of short time fluctuations. Velocity is the time integral of the acceleration, therefore, by adding random short time fluctuations of the acceleration, one finds the observed quasi-Gaussian velocity field.

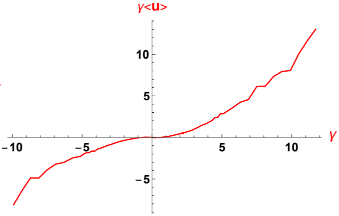
Scaling relations : $u^3 = \Gamma\gamma$ or $u\gamma \sim \epsilon$?



units: rms of the fluctuations.
Scalings Leray/circulation: $u^3 = \Gamma\gamma$;
Scaling Kolmogorov $u\gamma \sim \epsilon$: invalid

Circulation scaling vs Sedov-Taylor scaling vs Kolmogorov scaling





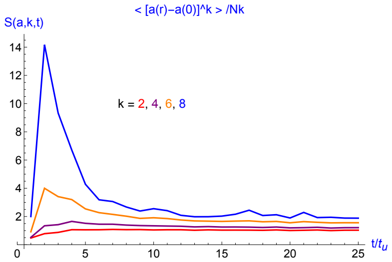
Scalings / circulation (left red): $u^3 \sim \Gamma\gamma$
Scalings / energy: Sedov-Taylor (left blue) $u^8 \sim E\gamma^3$;
Kolmogorov Scaling (right) $u\gamma \sim \epsilon$
Notice: Taylor frozen turbulence does not apply because the large velocity fluctuations and mean velocity are of the same order.

Sketch of a statistical theory based on the random occurrence of Leray singularities.1

$$\mathcal{S}_n(r) = \int \mathrm{d}q\, \nu_s(q) \int \mathrm{d}^3 r_0 \int \mathrm{d}t\, \left(a_s(r + r_0, t|q) - a_s(r_0, t|q)\right)^n$$

where $\mathcal{S}_n(r) = < (a(r + r_0) - a(r_0))^n >$ with $a_s(r, t|q)$ acceleration Leray-like solution singular at $t = r = 0$. Parameter q is for symmetries, and possibly a whole spectrum of solutions of Euler-Leray, $\nu_s(q)$ is the density of singularities in space-time. Two sources of dependence with respect to r: the phase-space part (i.e. the volume $\mathrm{d}^3 r_0 \mathrm{d}t$ at small r) and the singular dependence of a_s. If n is less than a critical value depending on the exponents of the Leray-like solution, $\mathcal{S}_n(r)$ tends smoothy to zero whereas it diverges at $r \to 0$ if n is larger than a critical value. This is in very good agreement with Modane's data. This sharp dependence of $\mathcal{S}_n(r)$ near $r = 0$ is a direct consequence of the existence of singular solutions in real turbulent flows.

Sketch of a statistical theory based on the random occurrence of Leray singularities.2



blue: $n = 8$
purple: $n = 4$
red: $n = 2$
Notice the very sharp difference between the behavior of $\mathcal{S}_n(r)$ for small r as n gets bigger.

Sketch of a statistical theory based on the random occurrence of Leray singularities.3

The explanation of this transition in behavior as n increases relies on the estimate of the contribution of singular events to $\mathcal{S}_n(r)$, assuming first that those events follow a Leray-like law of self-similarity and then that the solution of the Euler-Leray equation is linearly stable, or equivalently that Leray-like singularities have a nonzero basin of attraction in phase space of initial conditions (perhaps a too strong condition-see remarks below and coming arXiv paper). If one makes the first assumption, one finds that near $r = 0$:

$$\mathcal{S}_n(r) \sim r^{3+1/\beta - n(\alpha+1)/\beta}$$

The first contribution to the exponent comes from the volume of physical phase space $\mathrm{d}^3 r_0 \mathrm{d}t$, the other, proportional to n, from the divergence of the self-similar solution at $r = t = 0$. As n increases the exponent, as observed, changes from positive (decay of $\mathcal{S}_n(r)$ as r tends to zero) to negative (growth as r tends to zero, except for a possible round-off by viscosity very near $r = 0$).

Sketch of a statistical theory based on the random occurrence of Leray singularities.4

However, compared to the experimental values of the exponents the estimated exponents, when positive, are too big. This can be explained in three ways (not incompatible):
1) the parameter q related to the dilation invariance of the Euler equation depends on time τ and ultimately on viscosity, which amounts to add a contribution to u_s decaying like a power of τ. This takes into account that at very short distances viscosity becomes relevant and could explain why the Euler-exponent overestimates the growth of $\mathcal{S}_n(r)$ at small r.
2) α could belong to a continuous spectrum and takes negative values. This is consistent with the fact that the short distance distance behavior of the structure fundtion is dominated by the smaller values of the coefficient of n in the law for $\mathcal{S}_n(r)$. This goes against conservation of energy by Euler-Leray equations that gives the exponents their Sedov-Taylor value.
Point 3 on next slide

Sketch of a statistical theory based on the random occurrence of Leray singularities.5

3) After the singularity the fluctuation remains big and so contributes to the structure function, including at r small.
Last point: $\mathcal{S}_n(r)$ tends (quickly) to a constant as a function of r as r increases because it is made of the contributions of statistically independent singularities. This implies short distance correlation of the acceleration originating from independent localized singularities.
Local singularities explain *three* non trivial features of $\mathcal{S}_n(r)$: its large and small distance behavior and its dependence with respect to n.
Last point: viscosity makes disappear the singularity with the circulation exponents (the ones of Leray), but this is not so clear with Sedov-Taylor exponents because the local Reynolds number *increases* near the singularity time. Possible evolution toward a singularity with viscosity included. Rounding of the singularities by mechanisms other than viscosity (Burnett higher order effects in the Enskog expansion with fractional derivatives- YP PhD thesis).

Thanks

Thank you for your attention!