Interpretation of measurements in ^{4}He superfluid turbulence
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Dynamics of the Density of quantized vortex lines in superfluid turbulence

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Outline

- Vinen Equation
- Equation of motion for VLD in inhomogeneous flow
- Numerical setup
- Results
- Summary

Vinen Equation

Vinen equation

$$d\mathcal{L}(t)/dt = \underbrace{\mathcal{P}(t)}_{production} - \underbrace{\mathcal{D}(t)}_{decay}$$

Hall and Vinen works (1956-1958)

Vinen: Proc. R. Soc. A **238**, 204 (1956)
Proc. R. Soc. A **242**, 493 (1957)
Proc. R. Soc. A **243**, 400 (1958)
Hall : Phil. Trans. A **250**, 359 (1957)

$$\mathcal{P}(t) = \alpha C_1 \mathcal{L}^{3/2} |V_{ns}| \qquad \mathcal{D}(t) = \alpha\kappa C_2 \mathcal{L}^2$$

$\mathcal{L}$ - vortex line density

$\boldsymbol{V}_{\mathrm{ns}} = \boldsymbol{V}_{\mathrm{n}} - \boldsymbol{V}_{\mathrm{s}}$ - counterflow velocity

α - mutual friction parameter

C_1 and C_2 - fitting parameters

κ -quantum of circulation, for ^{4}He: $\kappa = 9.97 \times 10^{-8} cm^2/sec$

Dimensional analysis

$$\mathcal{P} \Rightarrow \mathcal{P}_{cl} \;=\; \alpha\kappa\mathcal{L}^2 F(x)$$

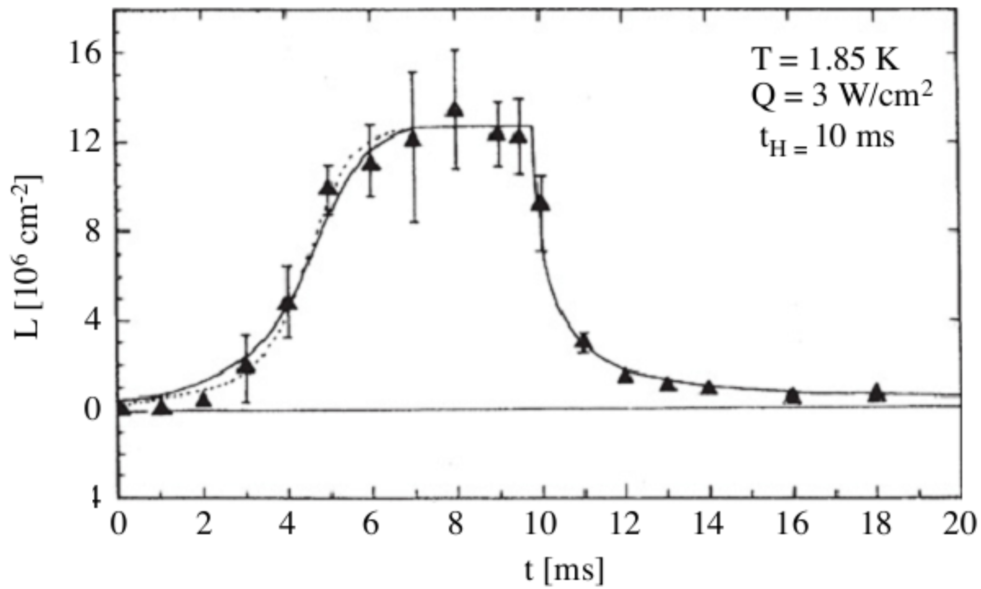
$$\mathcal{D} \Rightarrow \mathcal{D}_{cl} \;=\; \alpha\kappa\mathcal{L}^2 G(x)$$

$$x \equiv V_{\rm ns}^2/\kappa^2\mathcal{L}$$

$$\mathcal{P}_1 \;=\; \alpha C_1 \mathcal{L}^{3/2}|V_{ns}|$$

$$\mathcal{P}_2 \;=\; \alpha C_2 \mathcal{L} V_{ns}^2$$

$$\mathcal{P}_3 \;=\; \alpha C_3 \mathcal{L}^{1/2}|V_{ns}^3|/\kappa^2$$



Solution of Vinen equation with P-$_1$ (solid line) and P-$_2$ (dashed line) production terms

S.K. Nemirovskii et. al **PRB** 48, 1993

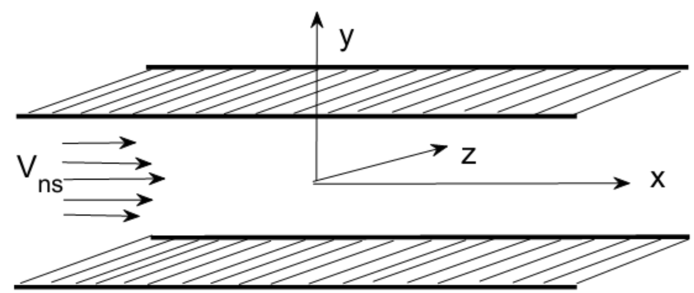
Generalization of Vinen's equation for nonhomogeneous case

$$\frac{\partial \mathcal{L}(r,t)}{\partial t} + \nabla \cdot \underbrace{\mathcal{J}(r,t)}_{flux} = \underbrace{\mathcal{P}(r,t)}_{production} - \underbrace{\mathcal{D}(r,t)}_{decay}$$

$\mathcal{J}(r,t)$?

$\mathcal{D}(r,t) \sim \mathcal{L}^2(r,t)$?

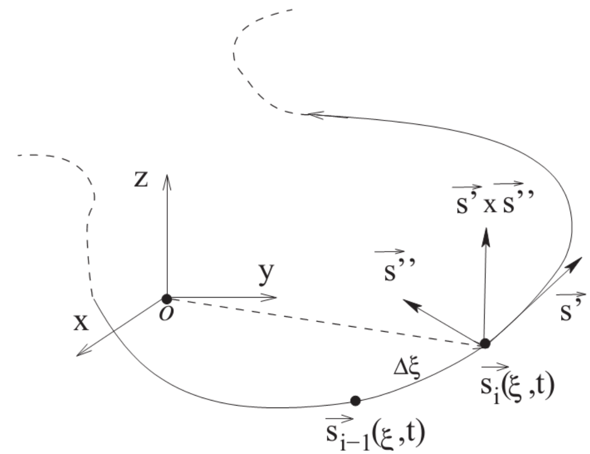
$\mathcal{P}(r,t) \sim V_{\mathrm{ns}}(r,t)\mathcal{L}^{3/2}(r,t)$?



Two dimensional channel counterflow

Equation of motion for single vortex line

$$\frac{d\boldsymbol{s}(\xi,t)}{dt} = \boldsymbol{V}^{\mathrm{s}}(\boldsymbol{s},t) + (\alpha - \alpha' \boldsymbol{s}' \times)\, \mathbf{s}' \times \boldsymbol{V}_{\mathrm{ns}}(\boldsymbol{s},t)$$



$$\boldsymbol{V}^{\mathrm{s}}(\boldsymbol{s},t) = \boldsymbol{V}_0^{\mathrm{s}} + \boldsymbol{V}_{\mathrm{BS}}(\boldsymbol{s})$$

$$\boldsymbol{V}_{\mathrm{BS}}(\boldsymbol{s}) = \frac{\kappa}{4\pi}\int_{\mathcal{C}} \frac{(\boldsymbol{s}-\boldsymbol{s}_1)\times \boldsymbol{d}s_1}{|\boldsymbol{s}-\boldsymbol{s}_1|^3} \Rightarrow \boldsymbol{V}_{\mathrm{LIA}}^{\mathrm{s}} + \boldsymbol{V}_{\mathrm{nl}}^{\mathrm{s}}(\boldsymbol{s})$$

local nonlocal

$$\boldsymbol{V}_{\mathrm{LIA}}^{\mathrm{s}}(\boldsymbol{s}) = \beta \boldsymbol{s}' \times \boldsymbol{s}'' \qquad \beta = -\frac{\kappa}{4\pi}\ln(a_0|s''|)$$

a_0 - vortex core radius, for ^{4}He $a_0 \approx 10^{-8} cm$

$$\boldsymbol{V}_{\mathrm{nl}}^{\mathrm{s}}(\boldsymbol{s}) = \frac{\kappa}{4\pi}\int_{\mathcal{C}'} \frac{(\boldsymbol{s}-\boldsymbol{s}_1)\times \boldsymbol{d}s_1}{|\boldsymbol{s}-\boldsymbol{s}_1|^3}$$

$\boldsymbol{V}_{\mathrm{ns}}^{0} = \boldsymbol{V}_{\mathrm{ns}} + \boldsymbol{V}_{\mathrm{LIA}}^{s}$ -counterflow velocity without local component

Vortex line is parametrized by radius vector $\boldsymbol{s}(\xi,t)$

ξ - arc length

$\boldsymbol{s}' = \dfrac{d\boldsymbol{s}}{d\xi}$ - unit vector in direction of vortex line

$\boldsymbol{s}'' = \dfrac{d^2\boldsymbol{s}}{d\xi^2}$ - vector of curvature

Equation of motion for vortex line segment length

$$d\delta\xi/(\delta\xi dt) = \boldsymbol{s}' \cdot d\boldsymbol{s}'/dt = \alpha \boldsymbol{V}_{\rm ns} \cdot (\boldsymbol{s}' \times \boldsymbol{s}'') + \boldsymbol{s}' \cdot \boldsymbol{V}_{\rm nl}^{\rm s}{}' - \alpha' \boldsymbol{s}'' \cdot \boldsymbol{V}_{\rm ns}$$

After integration

$$\frac{\partial \mathcal{L}(y,t)}{\partial t} + \frac{\partial \mathcal{J}(y,t)}{\partial y} = \mathcal{P}(y,t) - \mathcal{D}(y,t)$$

$$\mathcal{J}(y,t) = \frac{\alpha}{\Omega} \int s_{\rm z}' V_{\rm ns,x}^{0} d\xi$$

$$\mathcal{P}(y,t) = \frac{\alpha}{\Omega} \int \boldsymbol{V}_{\rm ns}^{0} \cdot (\boldsymbol{s}' \times \boldsymbol{s}'') d\xi$$

$$\mathcal{D}(y,t) = \frac{\alpha\beta}{\Omega} \int |s''|^2 d\xi$$

Analysis of the decay term

$$\mathcal{D}(y,t) = \frac{\alpha\beta}{\Omega}\int |\boldsymbol{s}''|^2 d\xi = \alpha\beta\mathcal{L}\tilde{S}^2$$

$\tilde{S}^2 \equiv \langle |s''|^2 \rangle$ -mean square curvature

$l \equiv \mathcal{L}^{-1/2}$ - intervortex distance

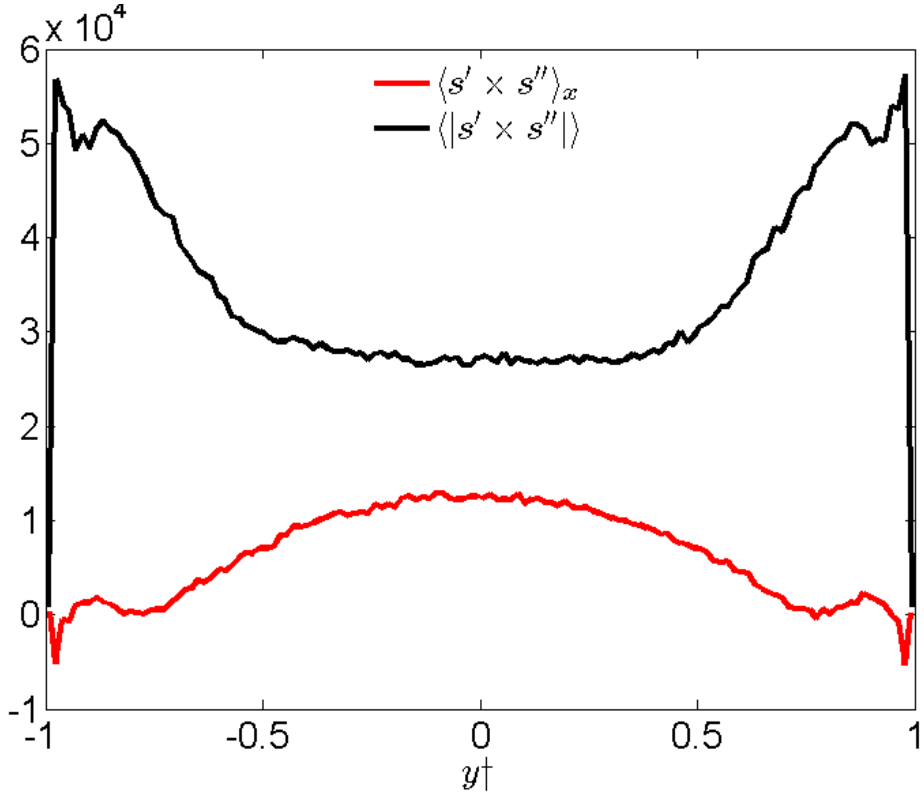
$$\tilde{S}^2 \sim \mathcal{L}$$

$$\mathcal{D}(y,t) = \alpha\beta C_{\mathrm{dec}}\mathcal{L}^2$$

Analysis of the production term

$$\mathcal{P}(y,t) = \frac{\alpha}{\Omega}\int \boldsymbol{V}^0_{\rm ns}\cdot(\boldsymbol{s}'\times\boldsymbol{s}'')d\xi$$

$$\mathcal{P}(y,t) = \alpha\mathcal{L}\boldsymbol{V}^0_{\rm ns}\cdot\langle\boldsymbol{s}'\times\boldsymbol{s}''\rangle$$

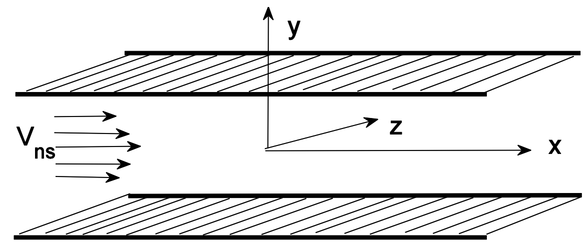


Analysis of the flux term

$$\mathcal{J}(y,t) = \frac{\alpha}{\Omega}\int s'_{\mathrm{z}} V^{0}_{\mathrm{ns,x}} d\xi = \frac{\alpha}{\kappa} V^{0}_{\mathrm{ns}} \frac{dV_{\mathrm{s}}}{dy}$$

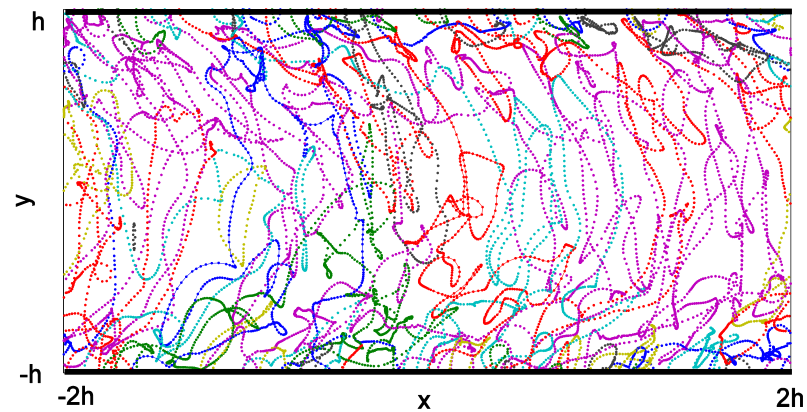
Numerical simulation

We consider counterflow in a planar channel



- Vortex filament model using full Biot-Savart calculations
- Computational domain 0.2x0.1x0.1 cm
- Periodic boundary conditions in x, z directions
  Solid walls with slip boundary conditions
- Line resolution $\Delta\xi = 1.6\times 10^{-3}$ cm
- Dissipative reconnection criterion
- T=1.6 K, $\alpha = 0.098$, $\alpha' = 0.016$.
- V_0^s calculated dynamically from the zero net mass flux condition

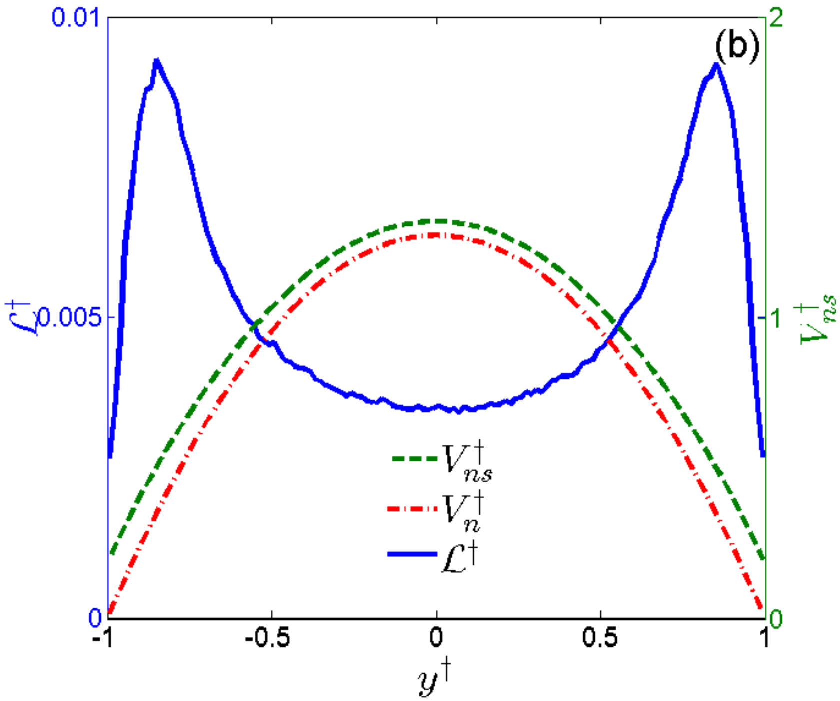
$$\rho_n \langle \mathbf{V}_n \rangle = \rho_s \langle V_0^s + \mathbf{V}_{BS} \rangle$$



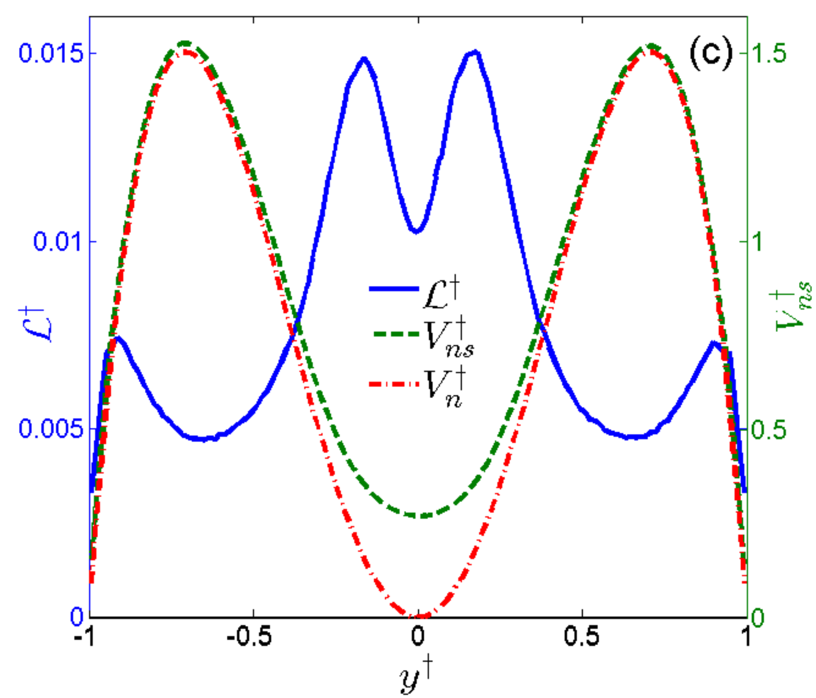
Two normal velocity profile types – parabolic and non-parabolic

Numerical results: profile of vortex line density

Parabolic profile



Non parabolic profile

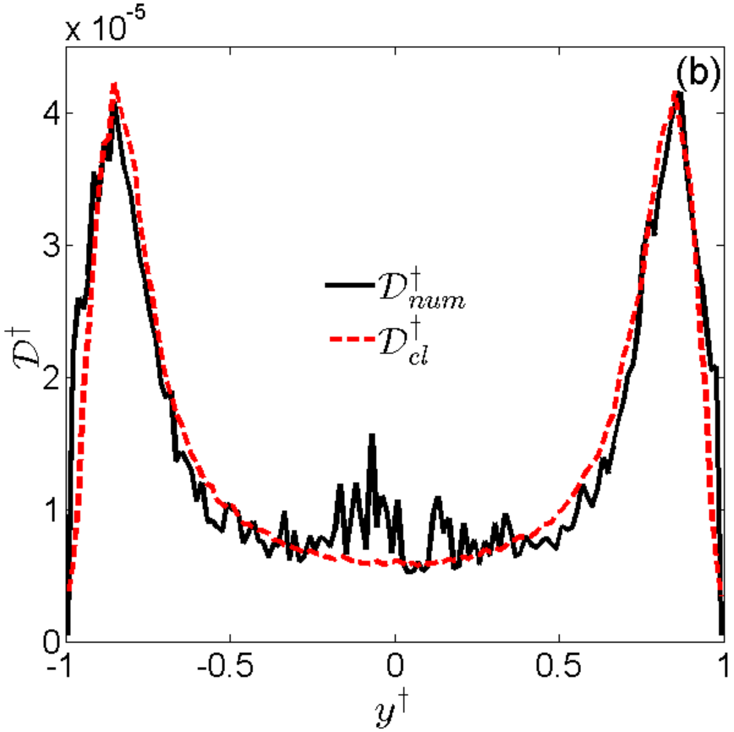


Normalized variables: $y^\dagger \equiv y/h;\ \mathcal{L}^\dagger \equiv \kappa^2 \mathcal{L}/\left\langle V_{\rm ns}^2 \right\rangle;\ V_{\rm n}^\dagger \equiv V_{\rm n}/\left\langle V_{\rm ns}^2 \right\rangle^{1/2}$

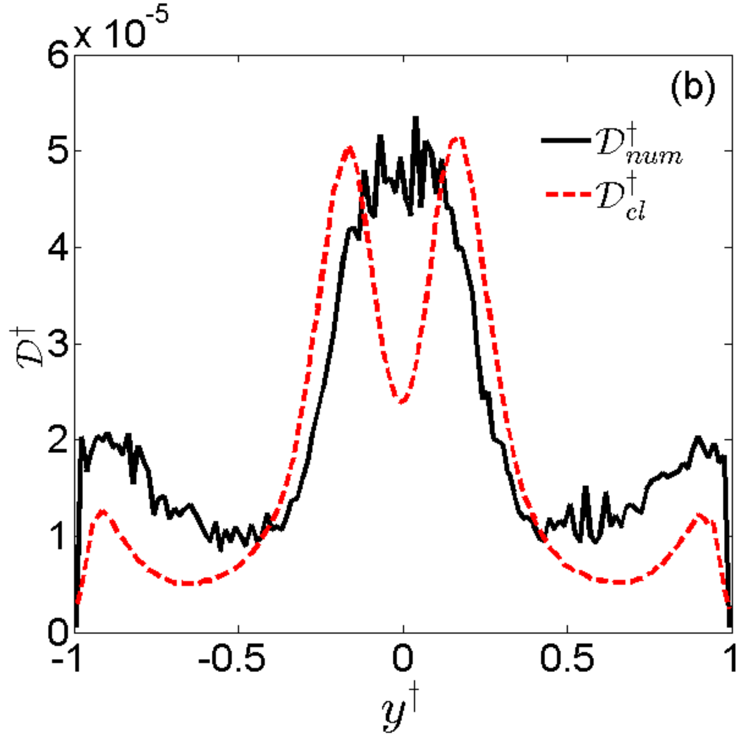
$$V_{\rm ns}^\dagger \equiv V_{\rm ns}/\left\langle V_{\rm ns}^2 \right\rangle^{1/2}$$

Numerical results: Decay

Parabolic profile



Non Parabolic profile

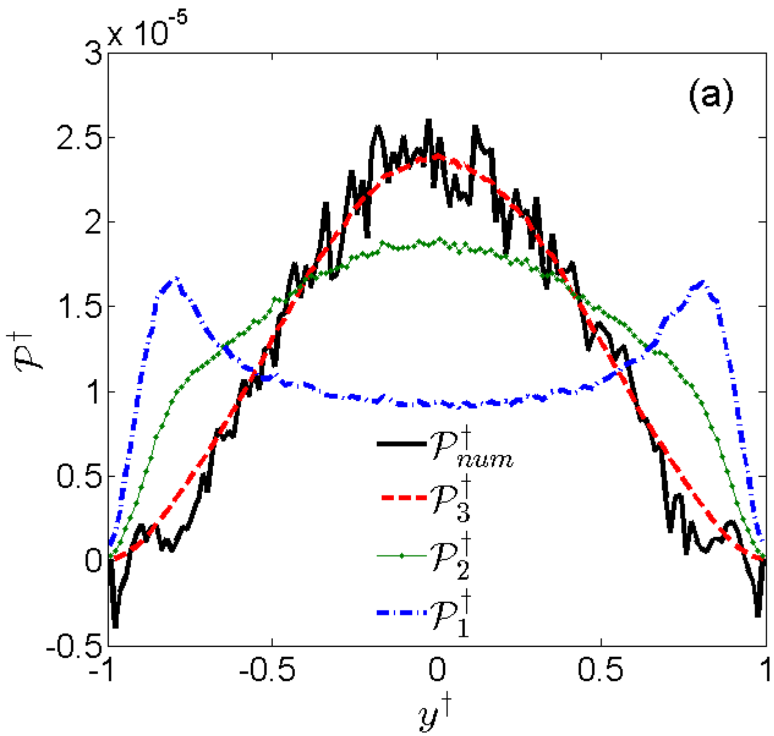


$$\mathcal{D}_{cl} = \alpha\beta C_{\mathrm{dec}}\mathcal{L}^2$$

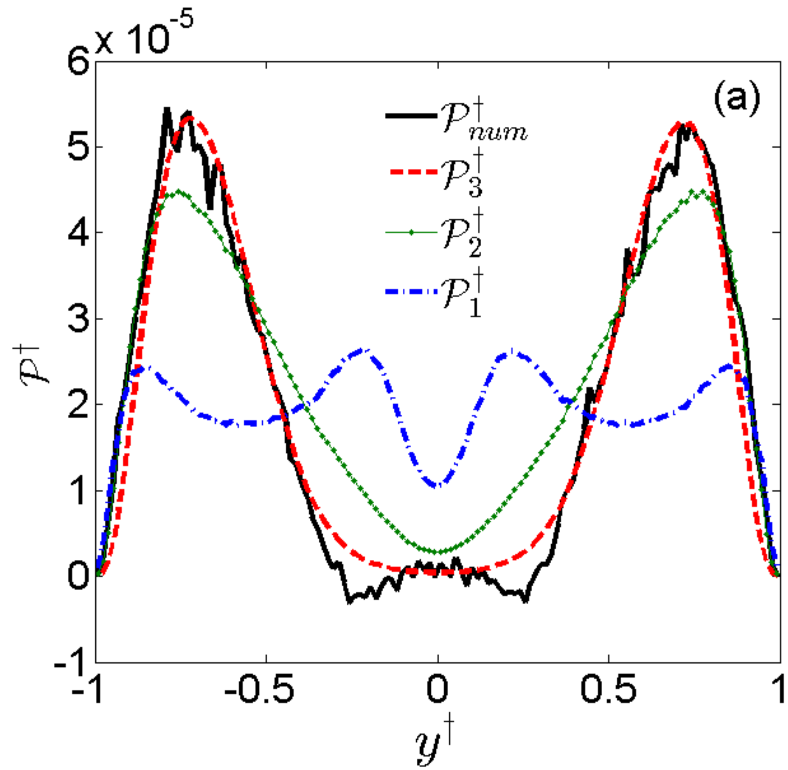
Normalized variables: $y^\dagger \equiv y/h;\ \ \mathcal{D}^\dagger \equiv \kappa^3\mathcal{D}/\left\langle V_{\mathrm{ns}}^2\right\rangle^2$

Numerical results: Production

Parabolic profile



Non parabolic profile

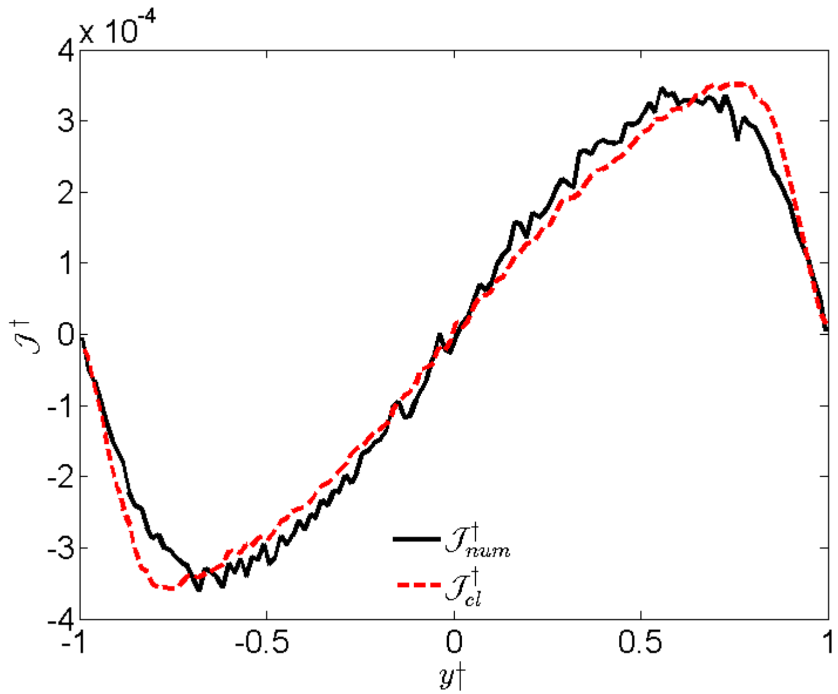


$$\mathcal{P}_1 = \alpha C_1 \mathcal{L}^{3/2} |V_{ns}|;\ \mathcal{P}_2 = \alpha C_2 \mathcal{L} V_{ns}^2;\ \mathcal{P}_3 = \alpha C_3 \mathcal{L}^{1/2} |V_{ns}^3| / \kappa^2.$$

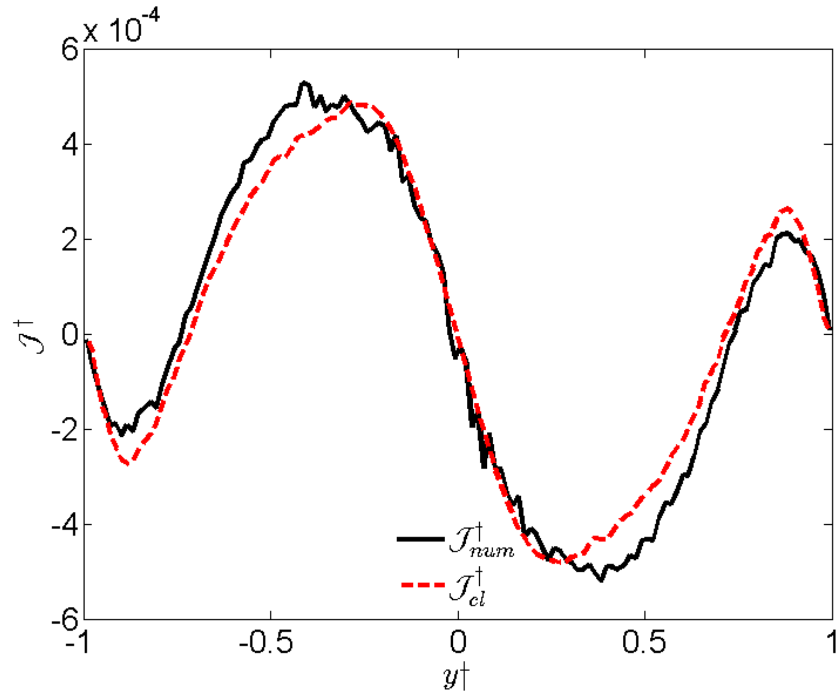
Normalized variables: $y^\dagger \equiv y/h; \quad \mathcal{P}^\dagger \equiv \kappa^3 \mathcal{P} / \langle V_{\mathrm{ns}}^2 \rangle^2$

Numerical results: Flux of the vortex line density

Parabolic profile



Non parabolic profile

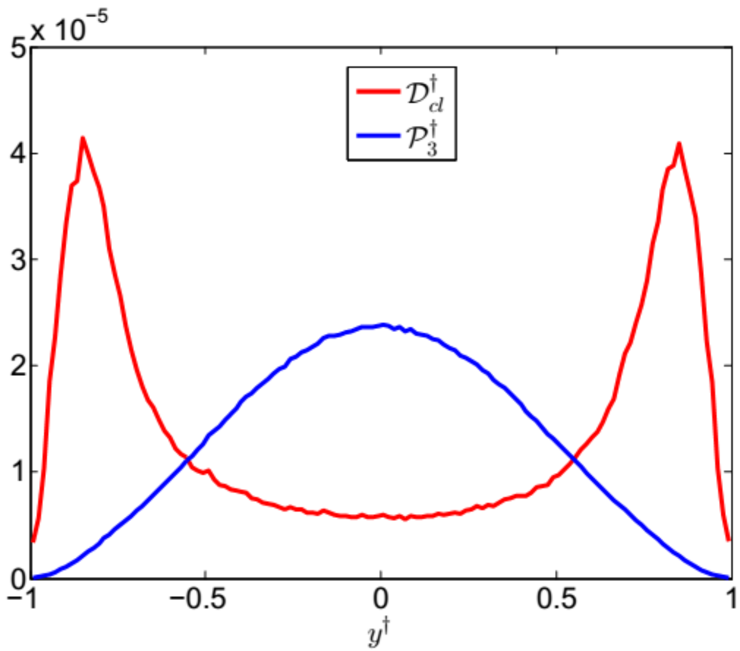


$$\mathcal{J}_{cl} = \frac{\alpha}{\kappa} V_{\mathrm{ns}}^0 \frac{dV_{\mathrm{s}}}{dy}$$

Normalized variables: $y^\dagger \equiv y/h; \quad \mathcal{J}^\dagger \equiv \kappa^2 \mathcal{J} / \langle V_{\mathrm{ns}}^2 \rangle^{3/2}$

Conclusion about equation for VLD

$$\frac{\partial \mathcal{L}}{\partial t} + \underbrace{\frac{\alpha}{\kappa}\frac{\partial}{\partial y}\left(V_{\rm ns}\frac{\partial V_s}{\partial y}\right)}_{flux} = \underbrace{\frac{\alpha C_3}{\kappa^2}\sqrt{\mathcal{L}}|V_{\rm ns}|^3}_{production} - \underbrace{\alpha\beta C_{dec}\mathcal{L}^2}_{decay}$$



Summary

- We have suggested the equation of motion for vortex line density in inhomogeneous channel counterflow
- We proposed closure relations for production, decay and VLD flux terms and verify them by direct numerical simulations using VFM in a plane channel
- We found quantitative agreement between the closure and the numerical results for parabolic and non-parabolic normal velocity profiles