

# Singularity turbulence

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# Motivation-context

- Role (and in fact existence) of singularities (or at least extreme events) in turbulence
- Extreme events (vorticity collapses) have been predicted and observed numerically (Siggia-Pumir 1987, Brachet et al 1992 for instance) and experimentally (Meneveau-Sreenivasan 1987). They are expected to be responsible of the intermittence in turbulence
- Difficult to handle in fluid flows: can we investigate a simpler model where we know the singularities (burgulence, Bec-Frisch 2000’).

# Focusing NLS

$$i \frac{\partial \psi}{\partial t} = -\frac{\alpha}{2} \nabla^2 \psi - g |\psi|^{2n} \psi$$

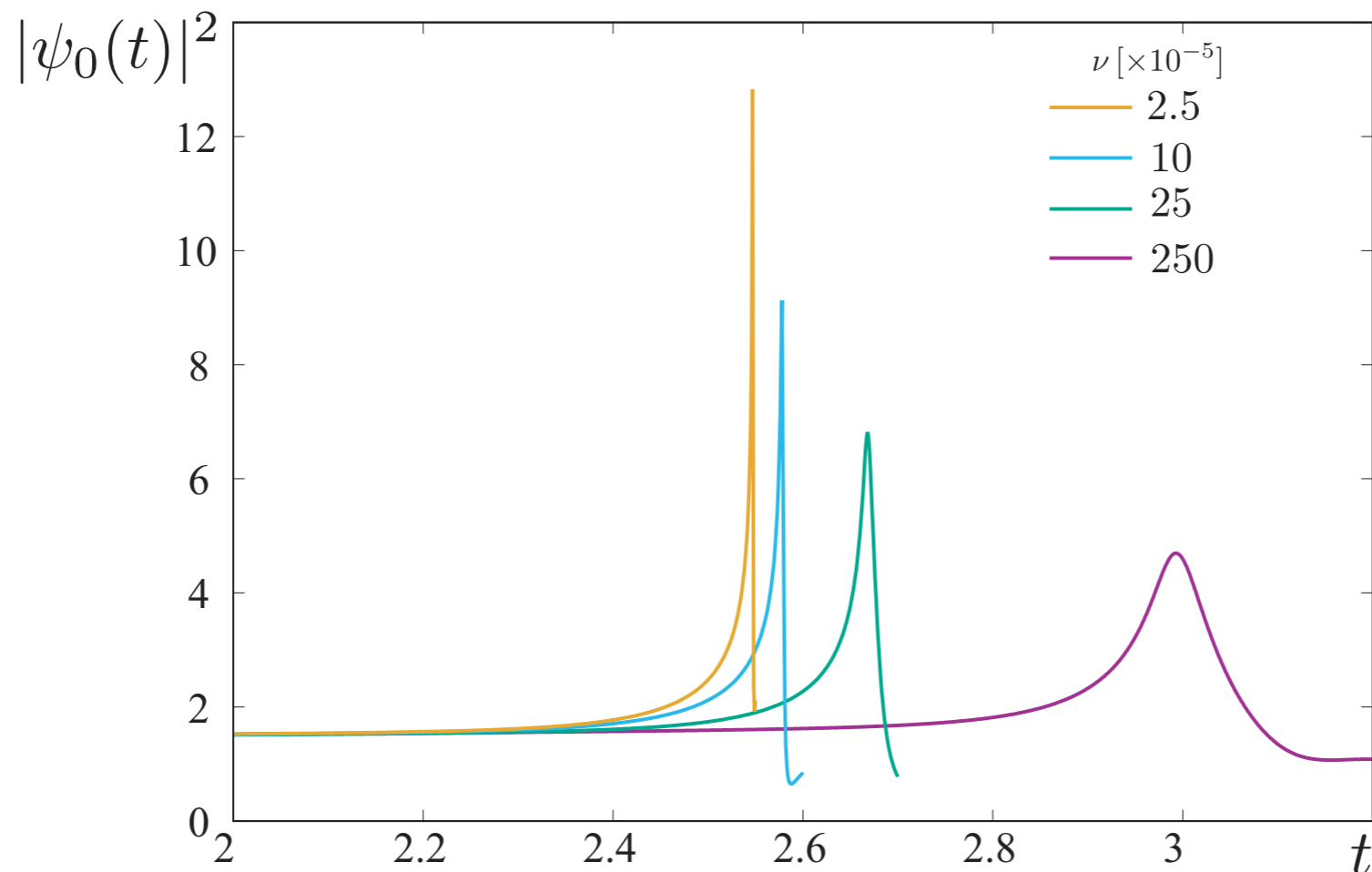
- $\psi(\mathbf{x}, t)$  is a complex field
- the NLS equation is a model for BEC, shallow-water and nonlinear optics.
- Hamiltonian structure and mass conservation

$$N = \int |\psi|^2 d\mathbf{x}$$

$$H = \int \left( \frac{\alpha}{2} |\nabla \psi|^2 - \frac{g}{n+1} |\psi|^{2(n+1)} \right) d\mathbf{x}$$

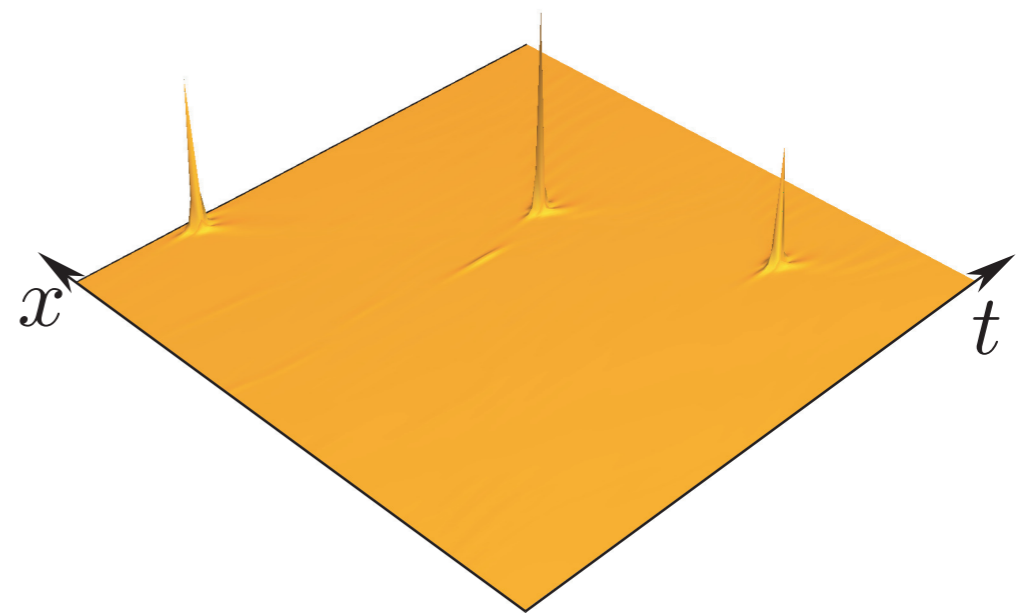
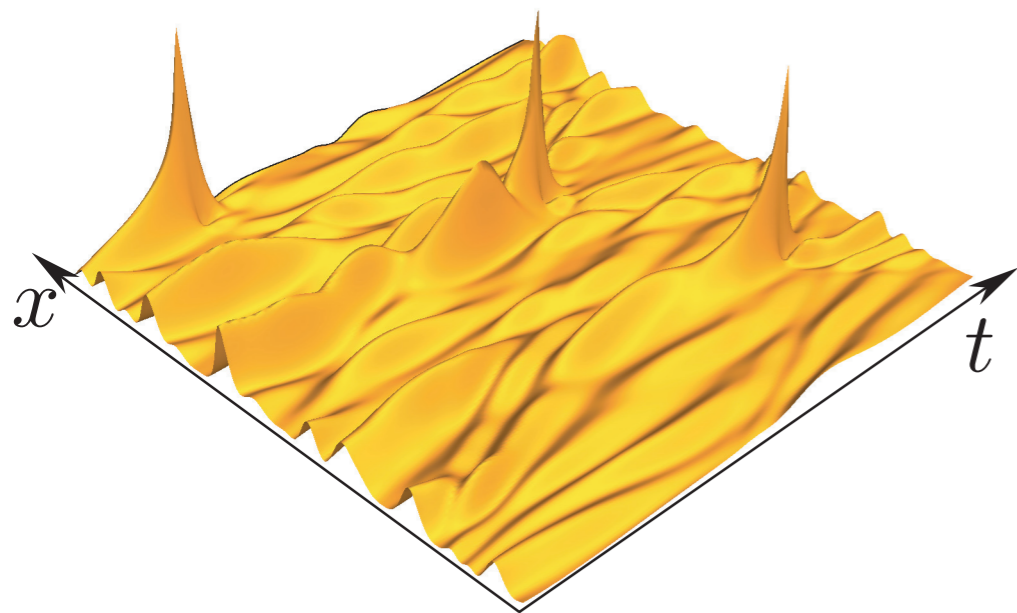
# Singularity in NLS

- for the focusing NLS ( $g=1$ ), the dynamics exhibits finite time singularities for  $nd>2$  (see for instance Le Mesurier et al 1988)
- this finite time singularity is suppressed in the presence of dissipation



$$i \frac{\partial \psi}{\partial t} = -\frac{\alpha}{2} \nabla^2 \psi - g |\psi|^{2n} \psi - \nu \Delta^2 \psi + f_{k_0}(\mathbf{x}, t).$$

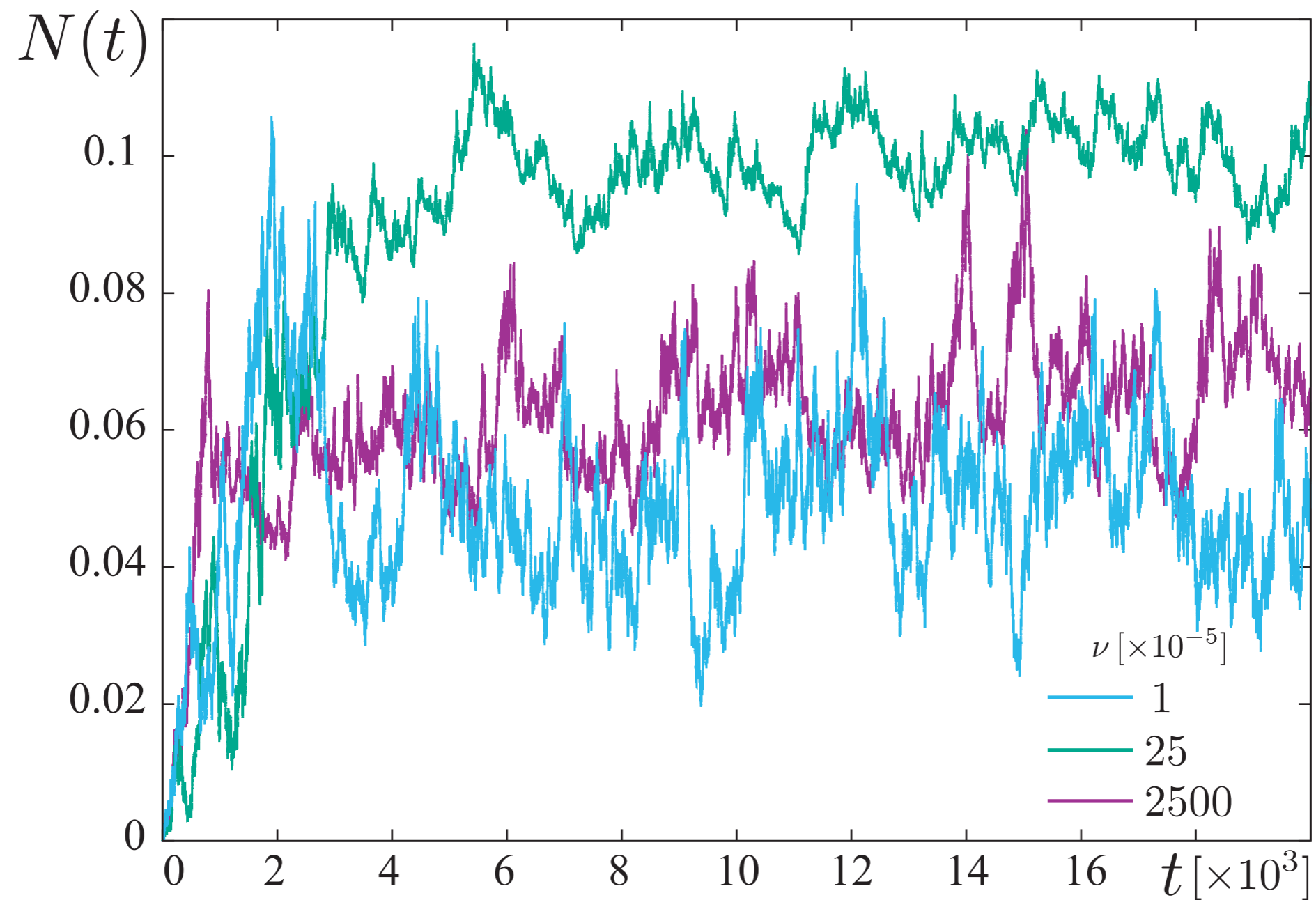
- with injection at large scale: turbulence of singularity or collapses (Dyachenko et al 1992)?
- important difference: here the mass (positive definite) is the pertinent quantity for the turbulence behavior
- wave turbulence (often observed in NLS equations) would suggest *inverse* cascade of mass and direct of energy
- focus here on the 1D case with  $n=3$  (work in progress in 2 and 3D)
- numerical simulations

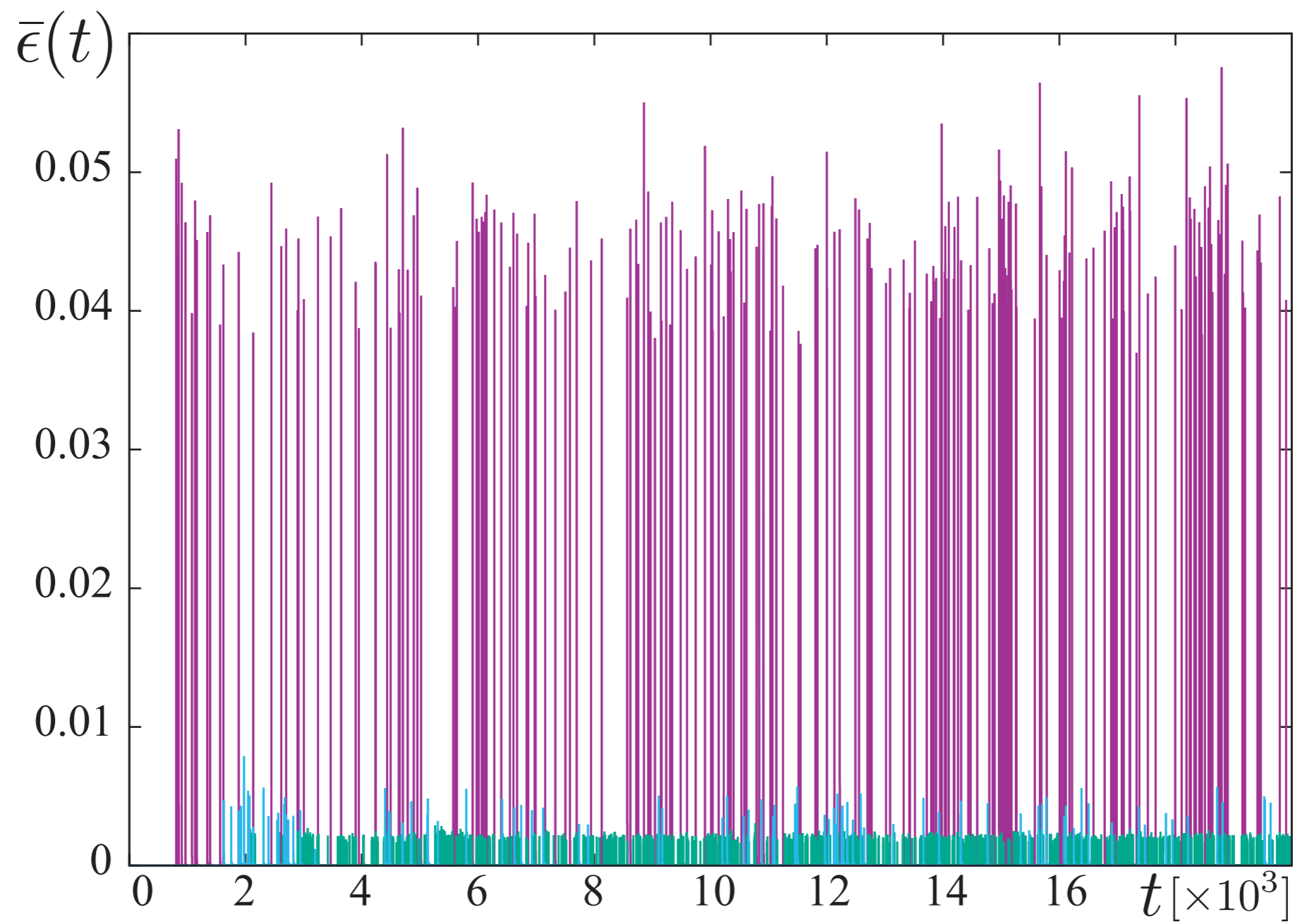


- « singularity » collapse or peak is followed by wave emission
- « dissipation » of mass is concentrated at short scale on the peaks:

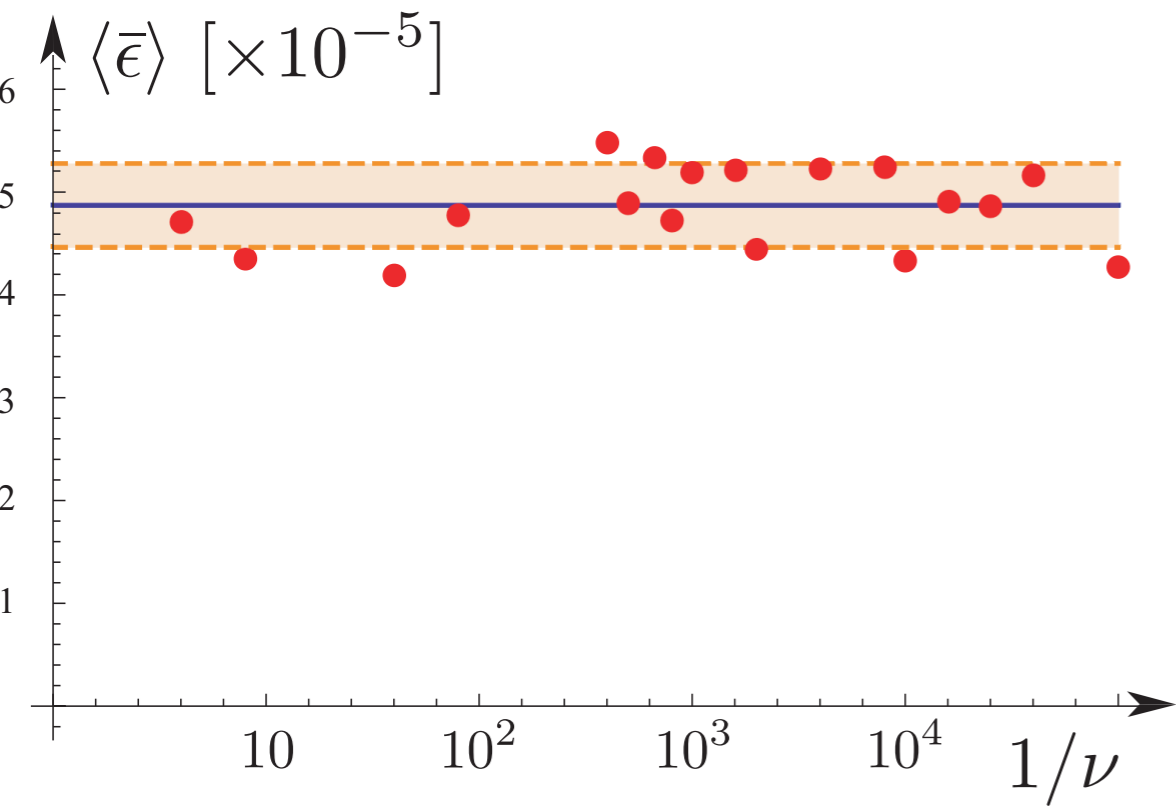
$$\frac{dN}{dt} = -2\nu \int |\Delta\psi|^2 d^D \mathbf{x} + i \int (\psi \bar{f}_{k_0} - \bar{\psi} f_{k_0}) d^D \mathbf{x}.$$

# Varying only the viscosity





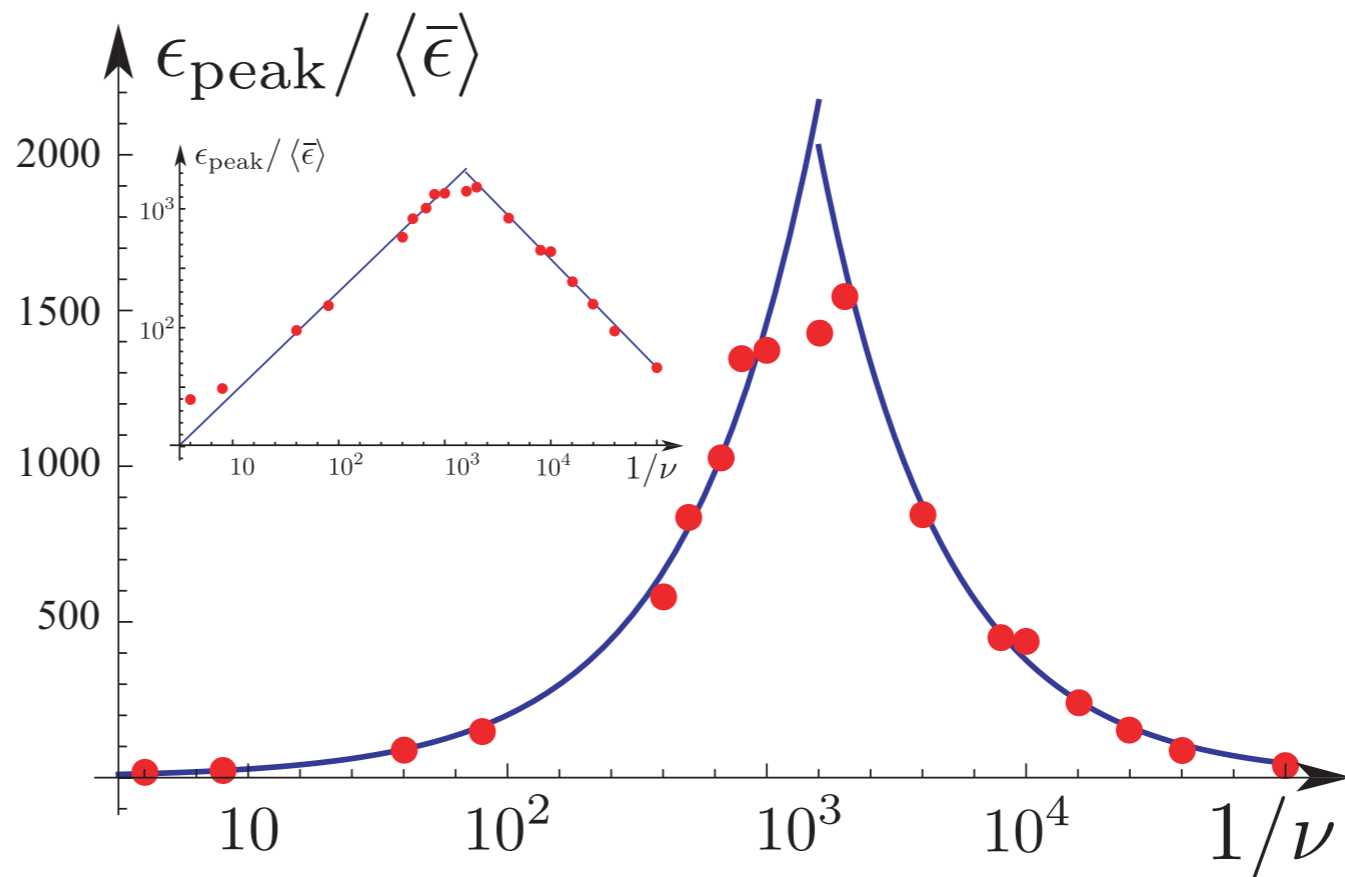




Anomalous  
dissipation?

Warning:

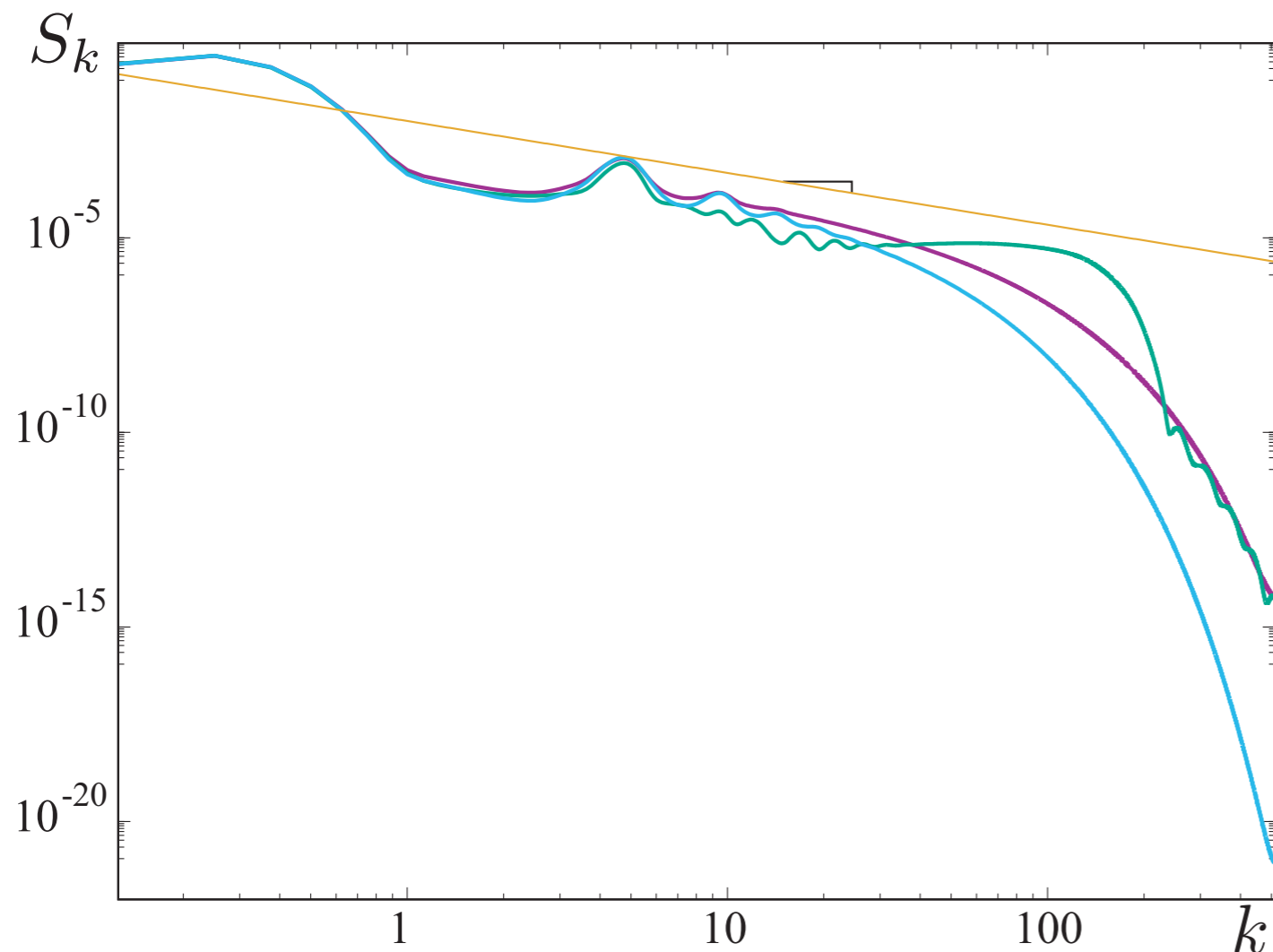
$$\nu \times |\Delta\psi|^2$$



# Spectrum

$$S_k(t) \equiv |\hat{\psi}_k|^2 + |\hat{\psi}_{-k}|^2 \qquad \frac{1}{L} \int |\psi|^2 d\mathbf{x} = \int |\hat{\psi}_k|^2 d\mathbf{k}$$

- Spectrum fluctuates at collapse



$$\frac{1}{t} \sim \frac{1}{x^2} \sim |\psi|^6$$

$$|\psi|^2 \sim x^{-2/3}$$

$$|\psi_k|^2 \sim k^{-4/3}$$

- spectrum of the self similar collapse

$$S_k \propto k^{-4/3}$$

- Transport equation for the spectrum

$$\frac{\partial S_k}{\partial t} = -\frac{\partial Q_k}{\partial k} - 2\nu k^4 S_k + F_k$$

- Need to consider averaged in time spectra for which we have

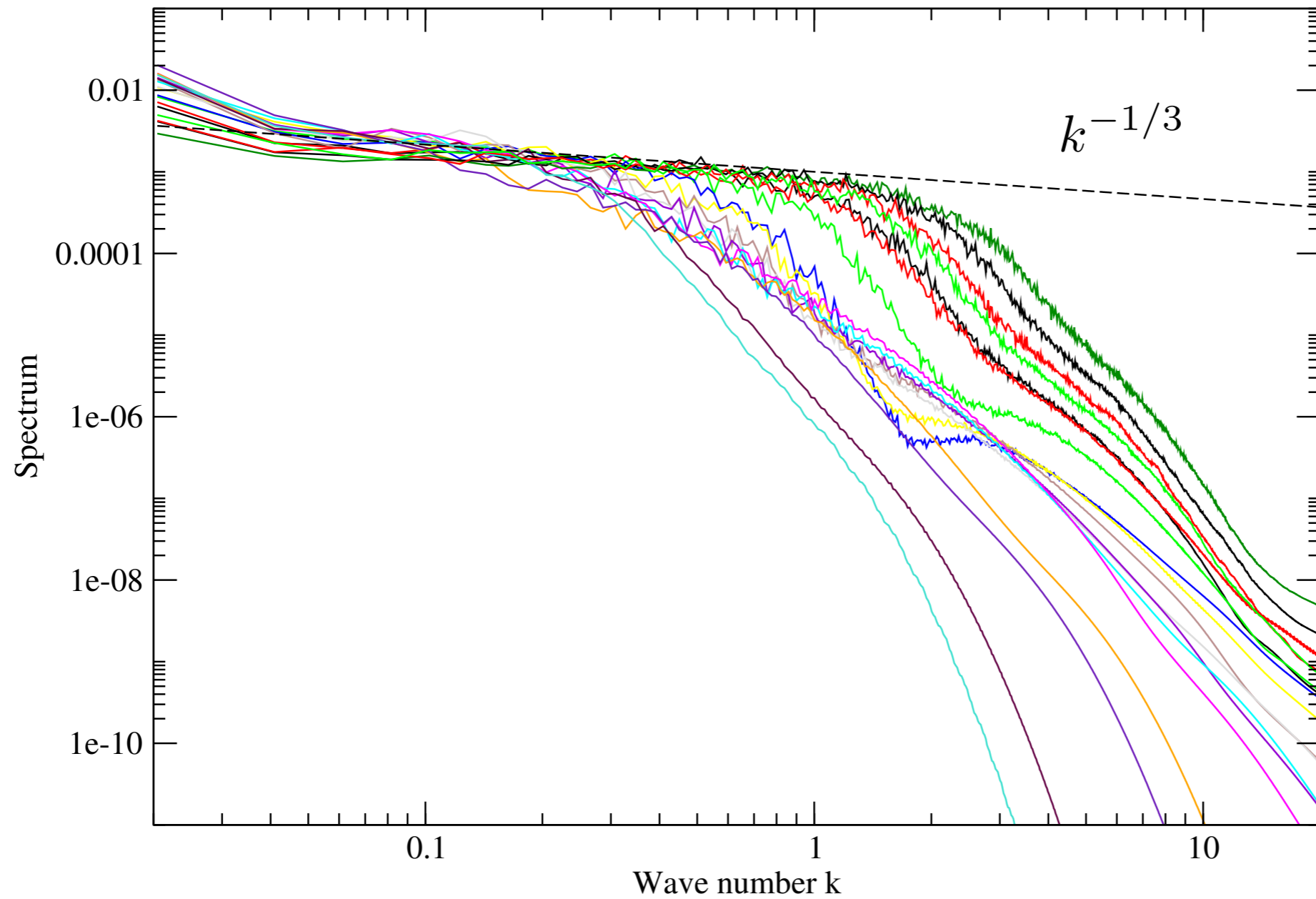
$$\frac{\langle \partial S_k \rangle}{\partial t} = 0$$

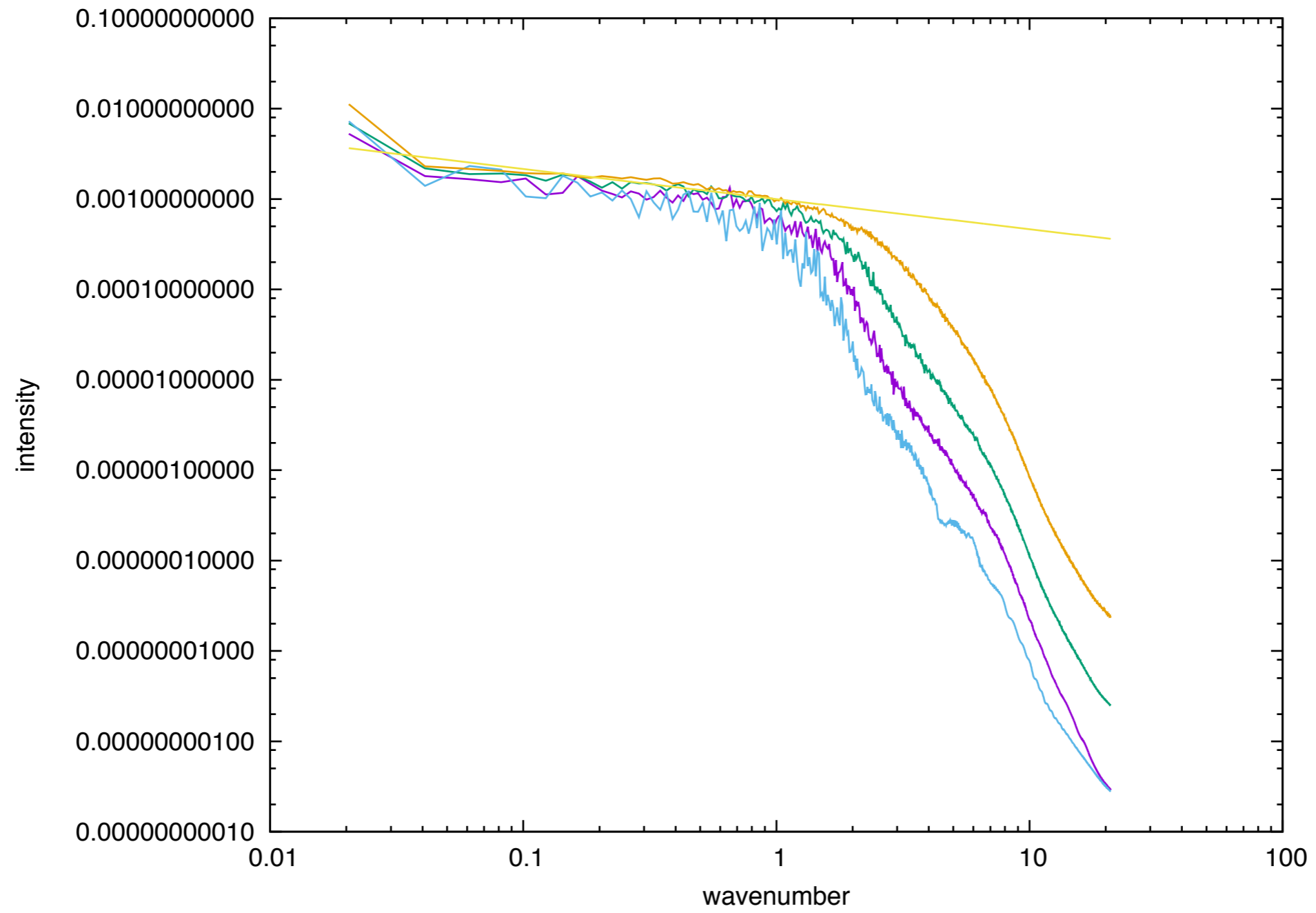
- In the inertial range we obtain also

$$\langle Q_k \rangle = -2\nu \int_k^\infty k^4 \langle S_k \rangle dk \equiv \langle \epsilon \rangle$$

-

Spectrum expands in  $k$  as the viscosity decreases. Its amplitude seems independent of the injection rate





## Kolmogorov-like scaling analysis

$$[S_k] = \rho l \quad [\epsilon] = \rho \tau^{-1} \quad [\alpha] = \ell^2 \tau^{-1} \quad [g] = \rho^{-3} \tau^{-1}$$

$$\langle S_k \rangle = \frac{\langle \bar{\epsilon} \rangle}{\alpha k^3} F \left( \frac{\alpha k^2}{(g \langle \epsilon \rangle^3)^{\frac{1}{4}}} \right)$$

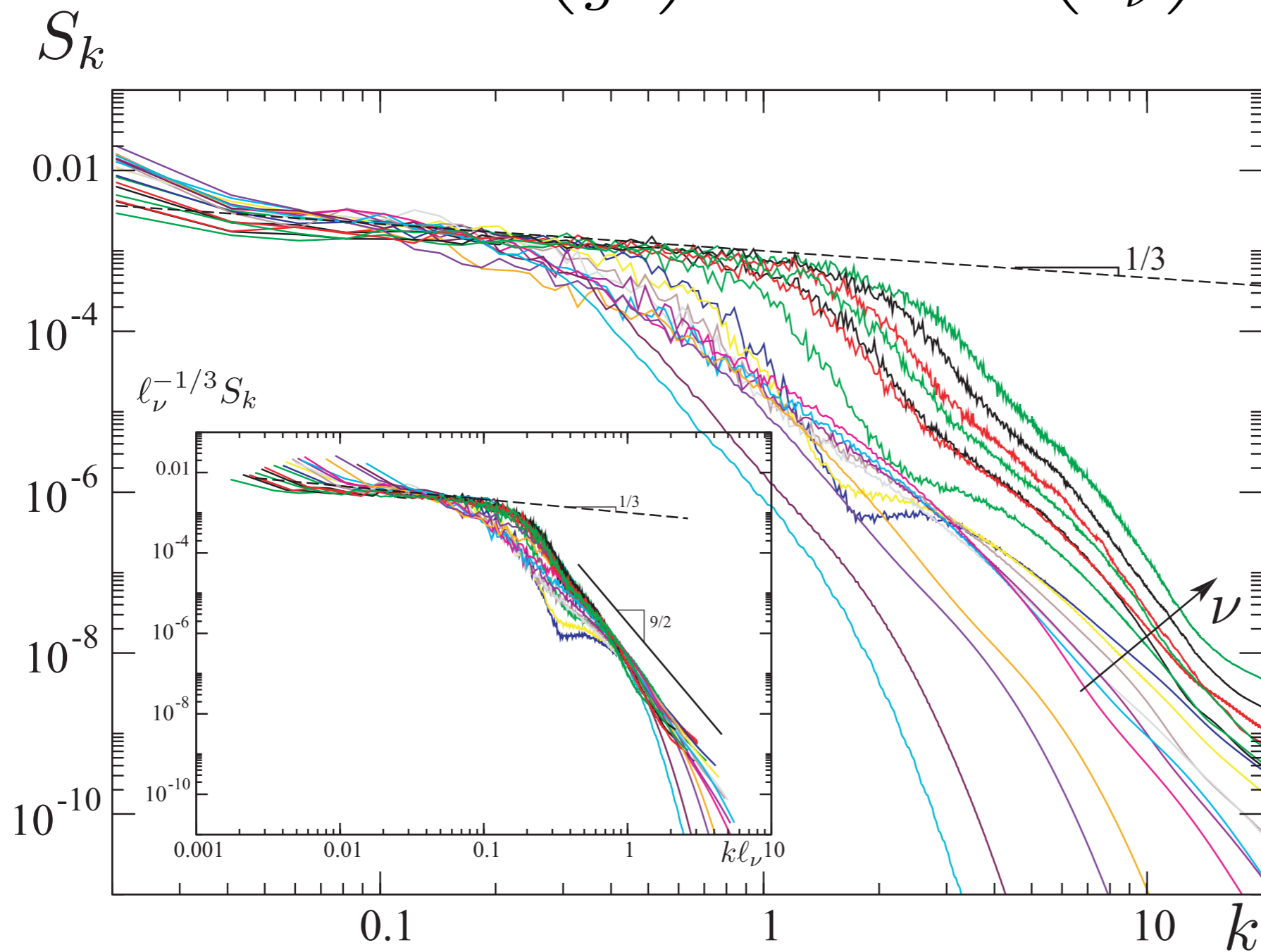
If we look for a solution independent of the injection rate

$$\langle S_k \rangle \propto \left( \frac{\alpha}{g^4} \right)^{1/3} k^{-1/3}$$

Kolmogorov scale  $l_\nu \sim \left( \frac{\alpha \nu^3}{g \bar{\epsilon}^3} \right)^{1/14} \quad k_\nu \sim \left( \frac{g \bar{\epsilon}^3}{\alpha \nu^3} \right)^{1/14}$

Suggest the following self-similar scaling for the spectrum

$$\langle S_k \rangle = \left( \frac{\alpha}{g^4} \right)^{1/3} k_\nu^{-1/3} G \left( \frac{k}{k_\nu} \right)$$



# Intermittency-structure functions

$$g_p(r) = \overline{|\psi(x+r) + \psi(x-r) - 2\psi(x)|^p}$$

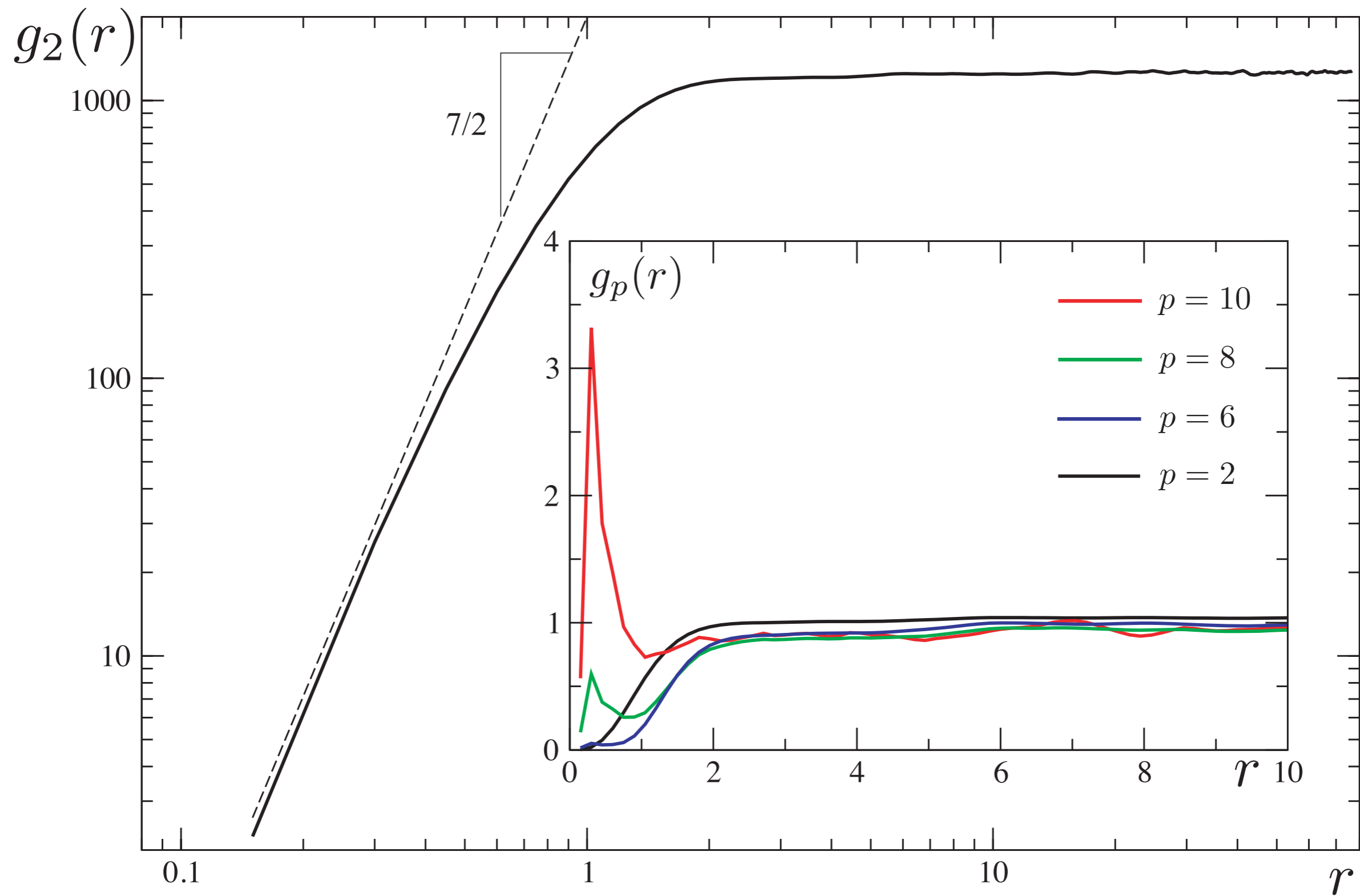
$p=2$  can be deduced from the spectrum scalings

$$g_2(r) \sim r^{7/2} \quad \text{at short scales}$$

$$g_2(r) \sim r^{-2/3} \quad \text{inertial range}$$

High  $p$ 's should witness the singularity at small scales





# Conclusion

- « singularity » mediated turbulence (in the spirit of « defect » mediated (Couillet, Gil & Lega 1989) is observed (singularity cured by viscosity) in a version of the focusing NLS
- simple model where singularity in the inviscid limit is known. Mass « cascade »
- dissipation of mass concentrated in the collapses
- Kolmogorov like spectra are observed (needs additional condition for the exponent, different that those of the collapse and of the WTT)
- Exhibits intermittency

Thanks! Questions?

