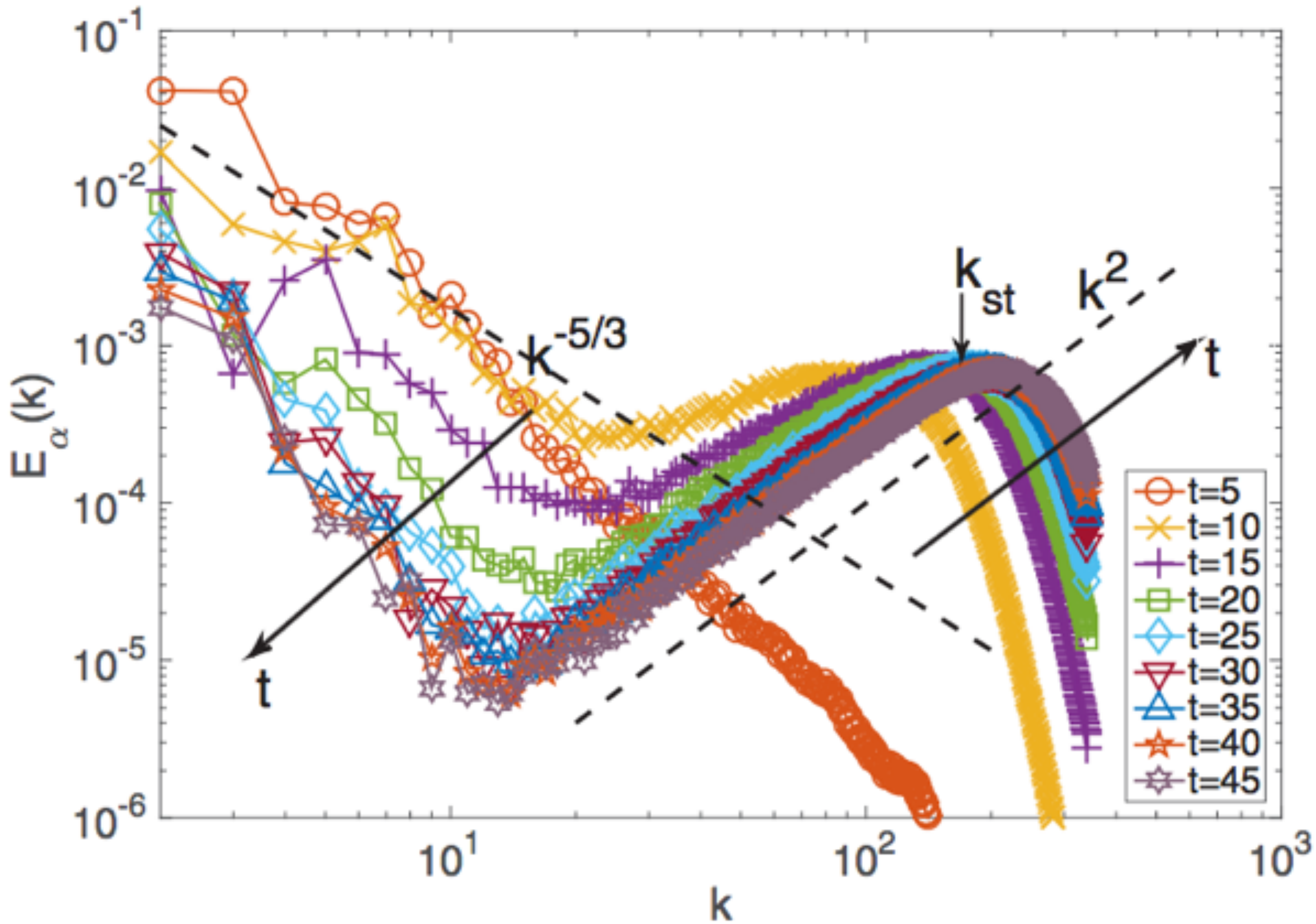
Self-truncation and scaling in Euler-Voigt-α and related fluid models



Giuseppe Di Molfetta, Giorgio Krstlulovic, Marc Brachet (speaker)

Interpretation of measurements in superfluid turbulence

September 14th to September 18th 2015

CEA Saclay, L'Orme des Merisiers

Plan of Talk

- Euler equation and spectral truncation...
- Gross-Pitaevskii Equation and truncation
- Self-truncation in the GPE
- Classical model of self-truncation
- Conclusion

Classical hydrodynamics

Truncated Euler equation

PRL **95,** 264502 (2005) PHYSICAL REVIEW LETTERS week ending 31 DECEMBER 2005

Effective Dissipation and Turbulence in Spectrally Truncated Euler Flows

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(Received 21 October 2004; published 22 December 2005)

A new transient regime in the relaxation towards absolute equilibrium of the conservative and time-reversible 3D Euler equation with a high-wave-number spectral truncation is characterized. Large-scale dissipative effects, caused by the thermalized modes that spontaneously appear between a transition wave number and the maximum wave number, are calculated using fluctuation dissipation relations. The large-scale dynamics is found to be similar to that of high-Reynolds number Navier-Stokes equations and thus obeys (at least approximately) Kolmogorov scaling.

Truncated Euler equation

TD. LEE (Quart Appl Math 1952), RH. KRAICHNAN 1967-1973, C. Cichowlas et al. (PRL 2005), W. BOS and J. Bertoglio (Phys. Fluids 2005), Frisch et al. (PRL 2008), ...

Euler PDE:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\mathbf{u}(\mathbf{x}, t) = \sum_{\mathbf{k}} \hat{\mathbf{u}}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$\partial_t \hat{u}_\alpha(\mathbf{k}, t) = -\frac{i}{2} \mathcal{P}_{\alpha\beta\gamma}(\mathbf{k}) \sum_{\mathbf{p}} \hat{u}_\beta(\mathbf{p}, t) \hat{u}_\gamma(\mathbf{k} - \mathbf{p}, t)$$

where $\mathcal{P}_{\alpha\beta\gamma} = k_\beta P_{\alpha\gamma} + k_\gamma P_{\alpha\beta}$ with $P_{\alpha\beta} = \delta_{\alpha\beta} - k_\alpha k_\beta / k$

Truncated Euler equation

Conserved quantities

Energy

$$E = \frac{1}{(2\pi)^3} \int \frac{|\mathbf{u}(\mathbf{x})|^2}{2} d^3x = \sum_k E(k)$$

Helicity

$$H = \frac{1}{(2\pi)^3} \int \mathbf{u}(\mathbf{x}) \cdot \omega(\mathbf{x})\, d^3x = \sum_k H(k) \quad , \quad \omega = \nabla \times \mathbf{u}$$

H. Moffatt, J. Moreau in the 60's. Discovered 200 years after Euler work

$$E(k) = \sum_{k-\Delta k/2<|\mathbf{k}'|<k+\Delta k/2} \tfrac{1}{2}|\hat{\mathbf{u}}(\mathbf{k}',t)|^2$$

$$H(k) = \sum_{k-\Delta k/2<|\mathbf{k}'|<k+\Delta k/2} \hat{\mathbf{u}}(\mathbf{k}',t)\cdot\hat{\omega}(-\mathbf{k}',t)$$

Both Energy and Helicity are exactly conserved by the truncated dynamics

Kraichnan's Helical

Absolute Equilibrium

(J. FLuids Mech. 73)

$$\hat{\mathbf{u}}(\mathbf{k}) \sim e^{-\beta E - \alpha H}$$

Gaussian field

$$E(k) = \frac{k^2}{\beta} \frac{4\pi}{1 - \alpha^2 k^2/\beta^2} \sim k^2 \qquad H(k) = \frac{k^4 \alpha}{\beta^2} \frac{8\pi}{1 - \alpha^2 k^2/\beta^2} \sim k^4$$

For the case presented here: $\alpha^2 k_{\max}^2/\beta^2 \ll 1$

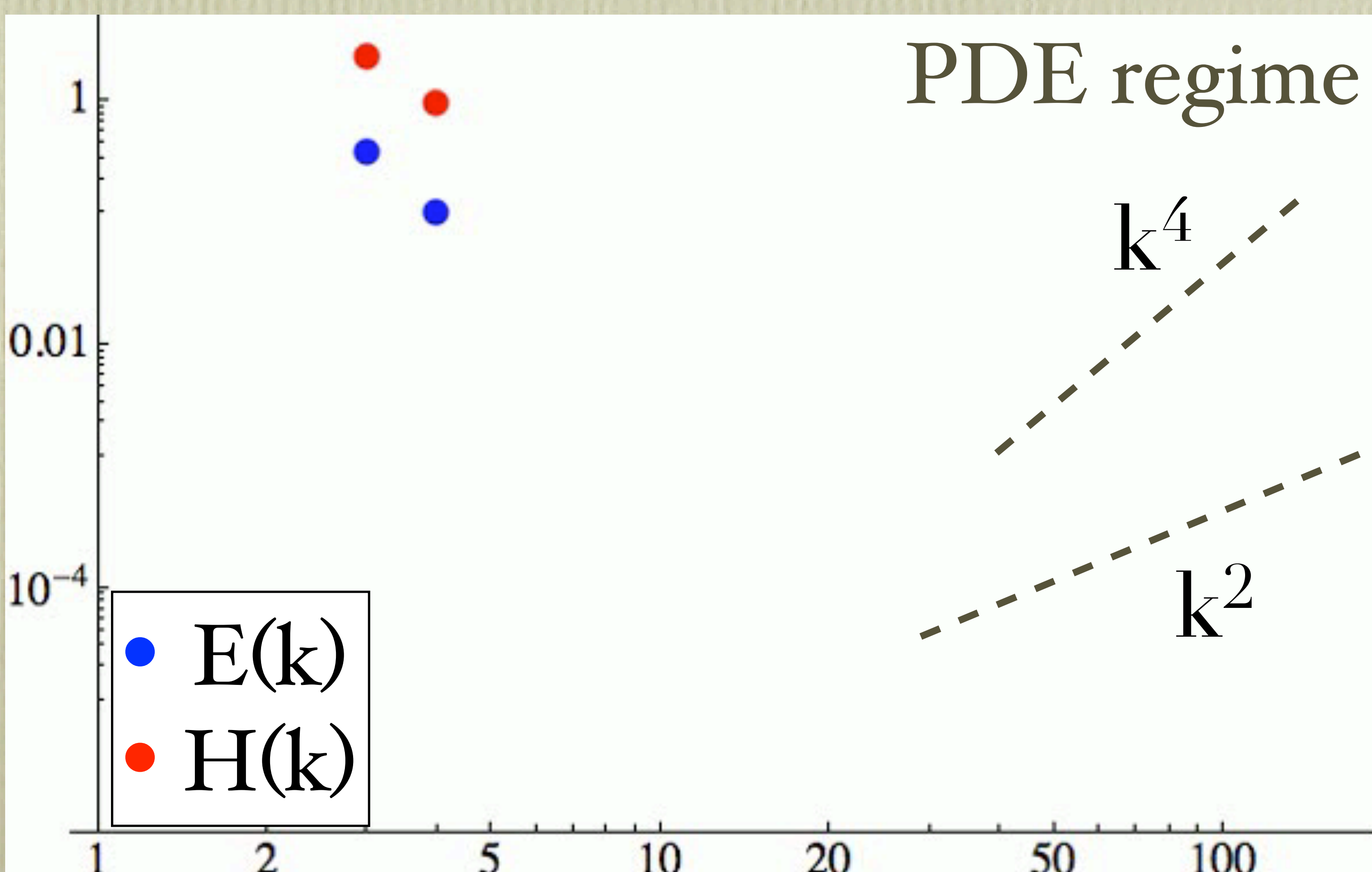
Numerical simulation ABC flow

Resolution of 512^3

$$\nabla \times \mathbf{u}_{\mathrm{ABC}}^{(k)} = \lambda_k \mathbf{u}_{\mathrm{ABC}}^{(k)}$$

G. Krstulovic, P. D. Mininni, M. E. Brachet and A. Pouquet, PRE 79(5) 056304, 2009

$$E(k) = \frac{k^2}{\beta} \frac{4\pi}{1 - \alpha^2 k^2/\beta^2} \sim k^2 \qquad H(k) = \frac{k^4 \alpha}{\beta^2} \frac{8\pi}{1 - \alpha^2 k^2/\beta^2} \sim k^4$$



Truncation effect

Partial Thermalization

Truncated Euler: basic facts

- Relaxation toward Kraichnan helical absolute equilibrium
- Transient mixed energy and helicity cascades
- Thermalized small-scales act as microworld providing an effective dissipation in the system

Superfluid hydrodynamics

Truncated Gross-Pitaevskii equation

PRL **106,** 115303 (2011) PHYSICAL REVIEW LETTERS week ending 18 MARCH 2011

Dispersive Bottleneck Delaying Thermalization of Turbulent Bose-Einstein Condensates

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(Received 26 July 2010; revised manuscript received 10 January 2011; published 16 March 2011)

A new mechanism of thermalization involving a direct energy cascade is obtained in the truncated Gross-Pitaevskii dynamics. A long transient with partial thermalization at small scales is observed before the system reaches equilibrium. Vortices are found to disappear as a prelude to final thermalization. A bottleneck that produces spontaneous effective self-truncation and delays thermalization is characterized when large dispersive effects are present at the truncation wave number. Order of magnitude estimates indicate that self-truncation takes place in turbulent Bose-Einstein condensates. This effect should also be present in classical hydrodynamics and models of turbulence.

Kolmorogov regime in the GPE

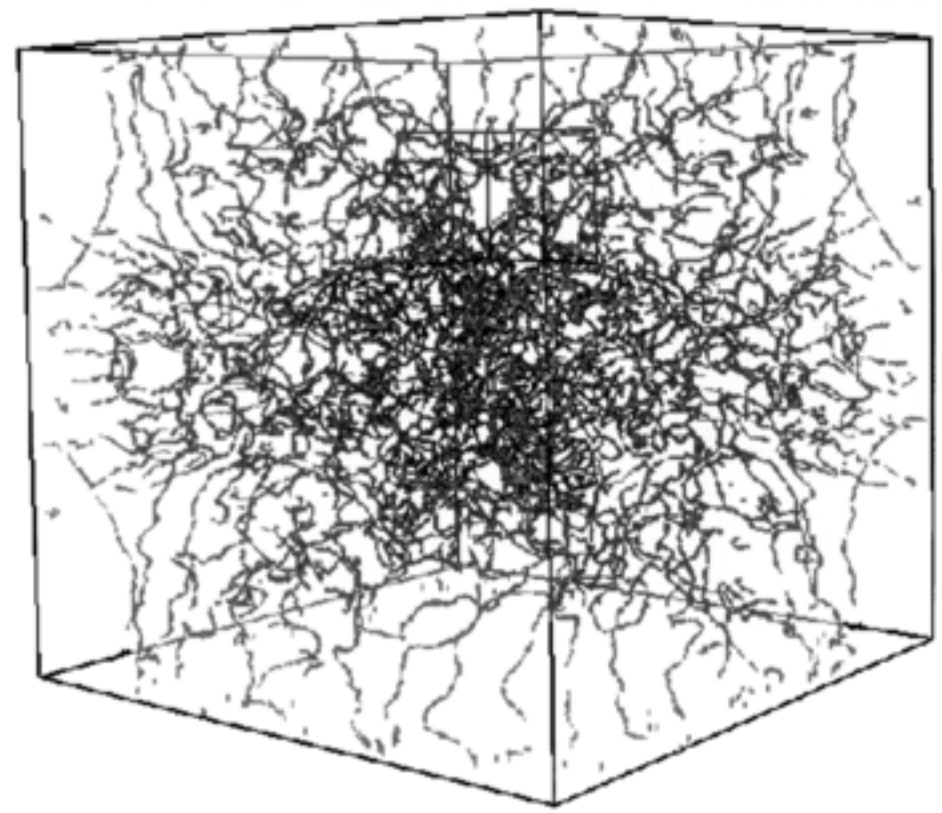


FIG. 5. Same visualization as in Fig. 1, but at time $t = 8$.

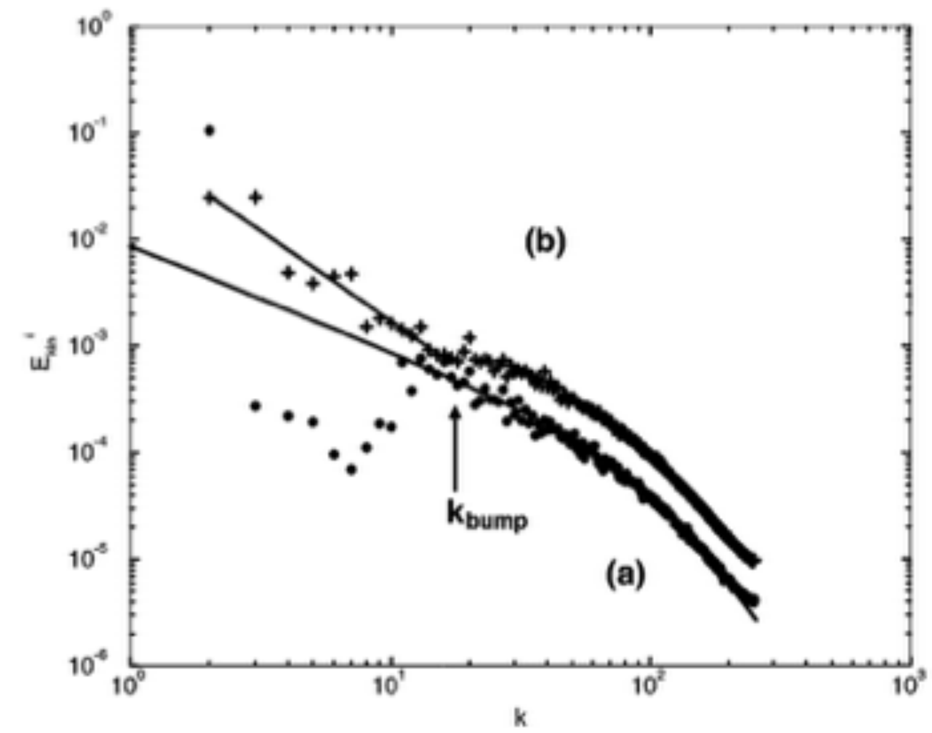


FIG. 2. Plot of the incompressible kinetic energy spectrum, $E^i_{\mathrm{kin}}(k)$. The bottom curve (a) (circles) corresponds to time $t = 0$ (same conditions as in Fig. 1). The spectrum of a single axisymmetric 2D vortex multiplied by $(l/2\pi) = 175$ is shown as the bottom solid line. The top curve (b) (plusses) corresponds to time $t = 5.5$. A least-square fit over the interval $2 \leq k \leq 16$ with a power law $E^i_{\mathrm{kin}}(k) = Ak^{-n}$ gives $n = 1.70$ (top solid line).

- K41 regime first found in the GPE 18 years ago:

  - C. Nore, M. Abid, and M. E. Brachet, *Phys. Rev. Lett. 78, 3896 (1997)*
  - C. Nore, M. Abid, and M. E. Brachet, *Phys. Fluids 9, 2644 (1997)*
  - M Kobayashi and M Tsubota. *Phy. Rev. Lett. 94(6):065302, Jan 2005.*
  - Yepez et al. *Phys. Rev. Lett. 103(8):084501, Aug 2009*
  - ....

Wave propagation

$$\psi = A_0 e^{-i\frac{\mu}{\hbar}t} + \delta\psi$$

Bogoliubov dispersion relation:

$$\omega(k) = \sqrt{\frac{g|A_0|^2}{m}k^2 + \frac{\hbar^2}{4m^2}k^4}.$$

$$\omega(k) = c\,k\sqrt{1 + \frac{1}{2}\xi^2 k^2}$$

Speed of sound $c = \sqrt{g|A_0|^2/m}$

Coherence length $\xi = \sqrt{\hbar^2/2m|A_\mathbf{0}|^2 g}.$

Important dimensionless parameter for TGPE

$$\xi k_{\mathrm{max}}$$

Amount of dispersion of thermal waves

Hydrodynamic description of GPE

$$\psi(\mathbf{x},t) = \sqrt{\frac{\rho(\mathbf{x},t)}{m}} \exp\left[i\frac{m}{\hbar}\phi(\mathbf{x},t)\right], \qquad \mathbf{v} = \boldsymbol{\nabla}\phi$$

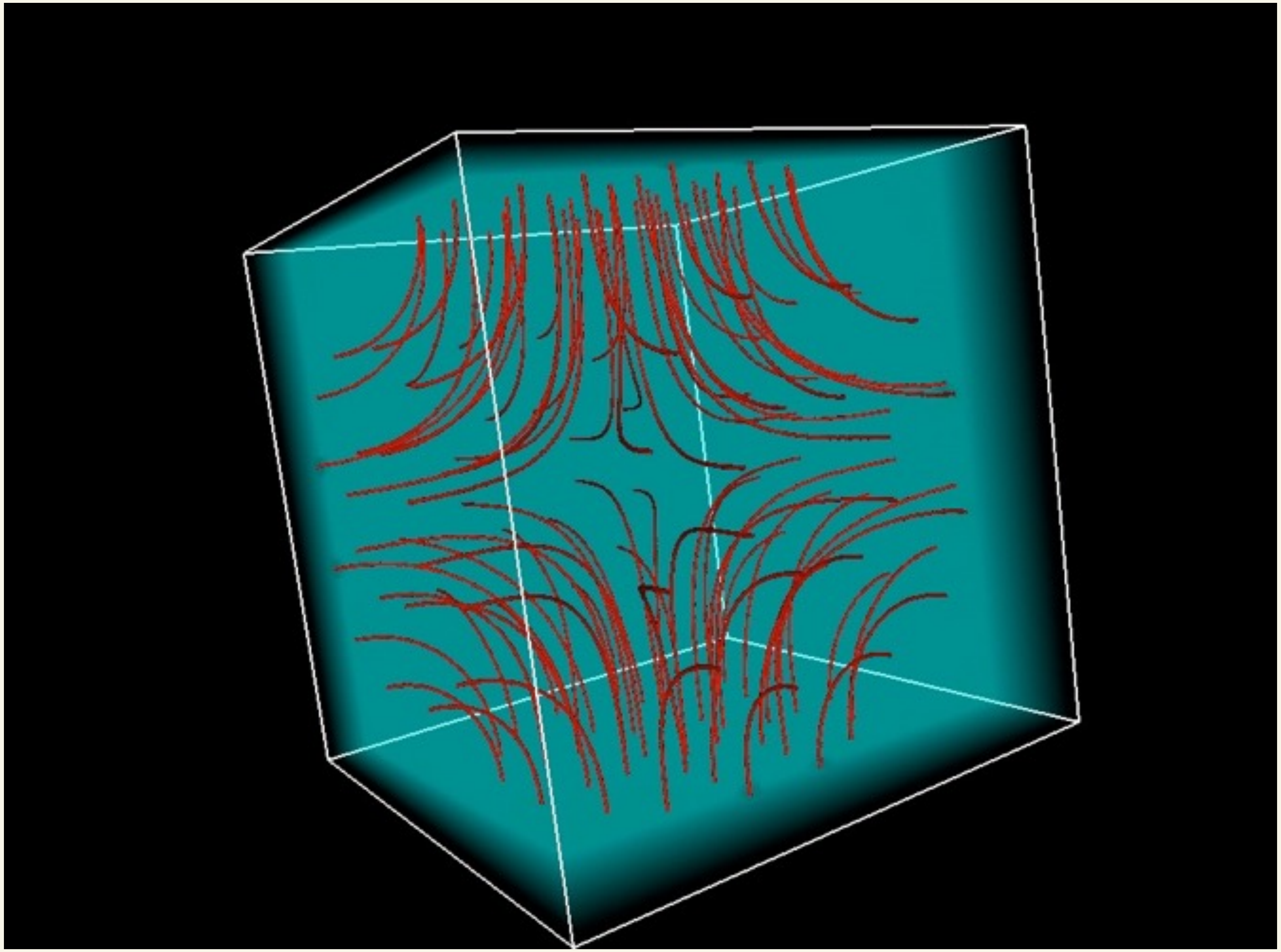
$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{\nabla}\,\phi) = 0, \qquad \frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 = c^2(1-\rho) + c^2\xi^2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}.$$

$$e_{\mathrm{tot}} = \frac{1}{V}\int d^3x \left[\frac{1}{2}(\sqrt{\rho}\mathbf{v})^2 + \frac{c^2}{2}(\rho-1)^2 + \frac{\hbar^2}{2m^2}(\nabla\sqrt{\rho})^2\right]$$

$$E_{\mathrm{tot}} = E_{\mathrm{Kin}} + E_{\mathrm{Int}} + E_{\mathrm{q}}$$

$$E_{\mathrm{Kin}} = E^{\mathrm{I}}_{\mathrm{Kin}} + E^{\mathrm{C}}_{\mathrm{Kin}}$$

Taylor-Green vortex



Energy transfer from incompressible kinetic energy to sound waves.

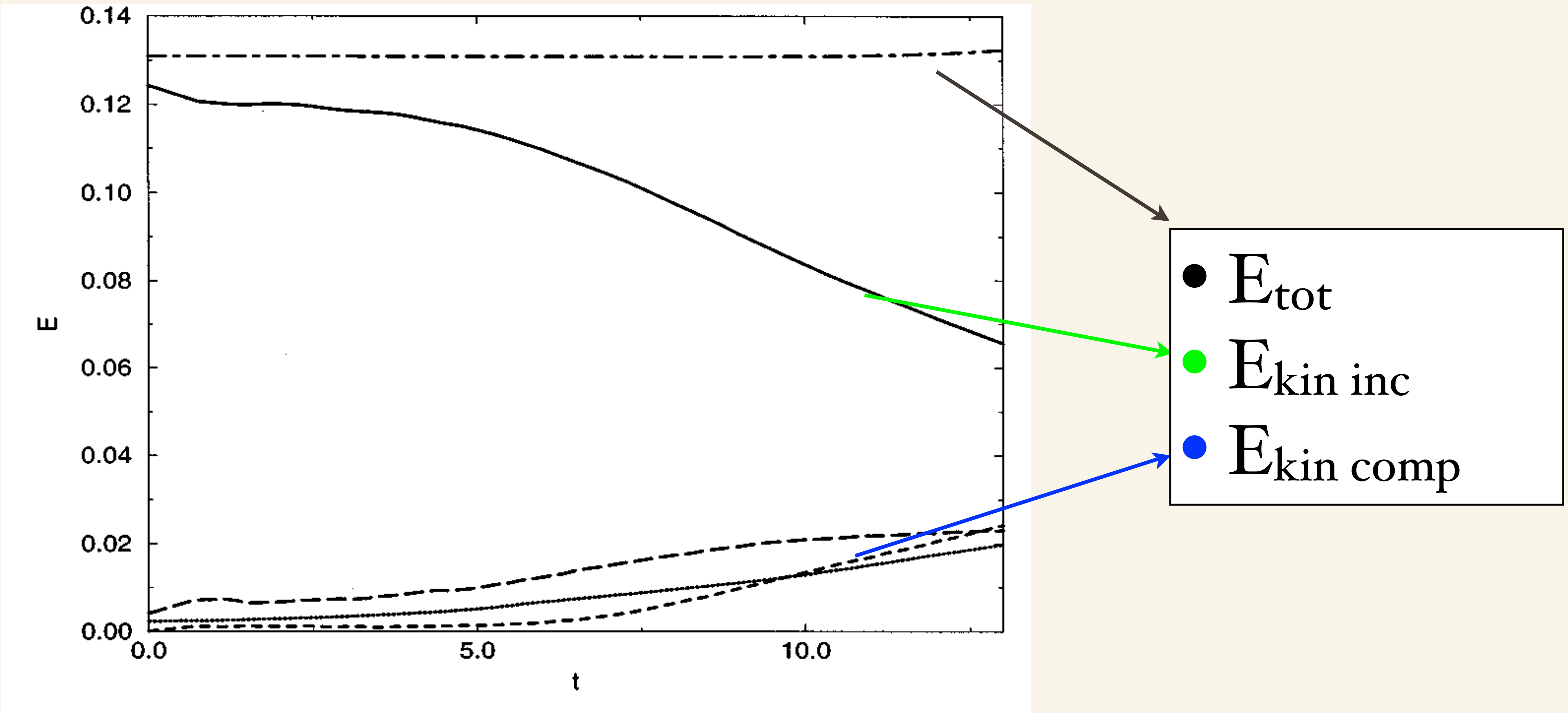


FIG. 13. Time evolution of total energy E_{tot} (dot-dashed), incompressible kinetic energy E^{i}_{kin} (solid), compressible kinetic energy E^{c}_{kin} (dotted), quantum energy E_q (dashed), and internal energy E_{int} (long–dashed) for run d. Note the transfer of energy from the incompressible part to the other contributions.

Nore et al. Phys. Fluids **9** (9), 1997, PRL 78 (30), 1997

Truncation of GPE

$$i\hbar\frac{\partial\psi}{\partial t} = \mathcal{P}_{\mathrm{G}}[-\frac{\hbar^2}{2m}\nabla^2\psi + g\mathcal{P}_{\mathrm{G}}[|\psi|^2]\psi]$$

$$H = \int d^3x\left(\frac{\hbar^2}{2m}|\nabla\psi|^2 + \frac{g}{2}[\mathcal{P}_{\mathrm{G}}|\psi|^2]^2\right).$$

$$\mathcal{P}_{\mathrm{G}}[\hat{\psi}_k] = \theta(k_{\mathrm{max}} - k)\hat{\psi}_k$$

Heaviside function

Description of BEC at finite temperature: Thermal fluctuations overwhelm quantum fluctuations

Conserved quantities

Energy, number of particles and momentum

$$
\begin{aligned}
H &= \int_V d^3x \left(\frac{\hbar^2}{2m} |\nabla \psi|^2 + \frac{g}{2} |\psi|^4 \right) \\
N &= \int_V |\psi|^2 \, d^3x \\
\mathbf{P} &= \int_V \frac{i\hbar}{2} \left(\psi \nabla \overline{\psi} - \overline{\psi} \nabla \psi \right) \, d^3x.
\end{aligned}
$$

Conservation laws are valid in the truncated system, if dealiasing is done carefully enough

Thermalized microcanonical states

Condensation transition in TGPE

It was previously known that

the k=0 mode of ψ vanishes at finite energy

MJ. Davis, SA. Morgan and K. Burnett
PRL **87**, (2001)

C. Connaughton, C. Josserand, A. Picozzi, Y. Pomeau and S. Rica. PRL **95**, 263901.(2005)
Düring et al. Physica D 2009, vol. 238

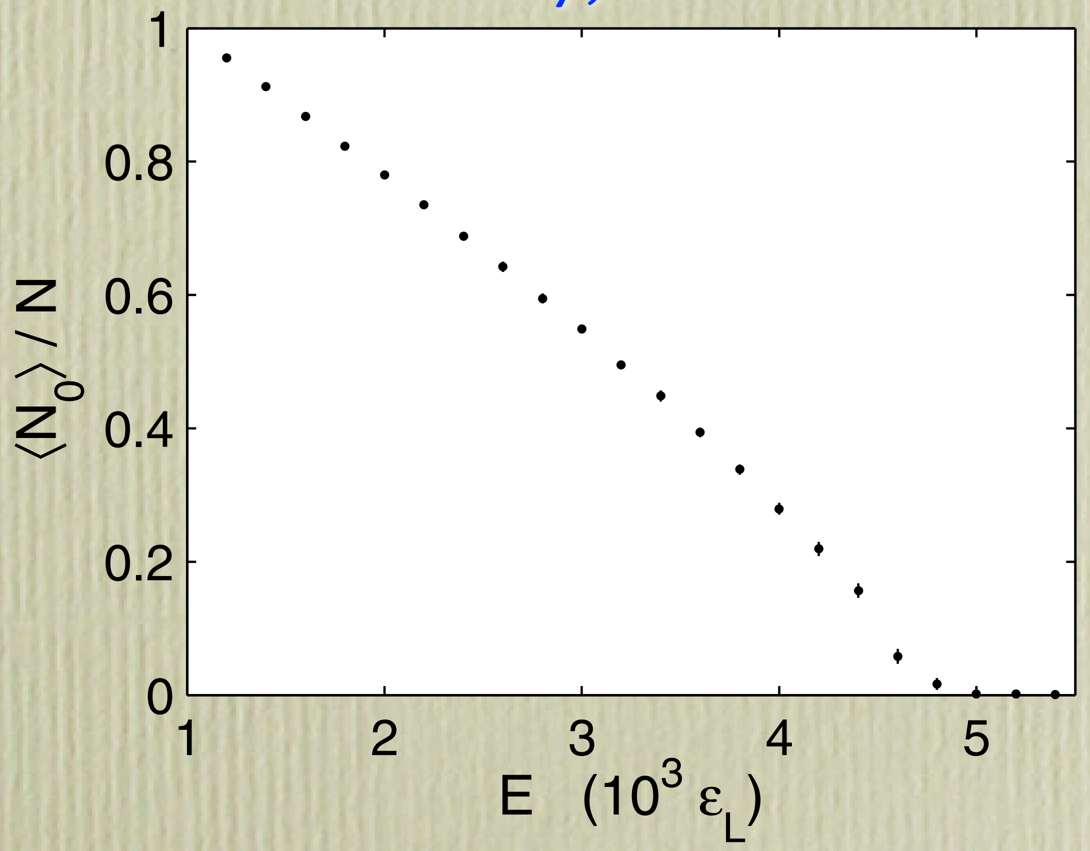


FIG. 1. Condensate fraction plotted against total energy after each individual simulation has reached equilibrium. The barely discernible vertical lines on each point indicate the magnitude of the fluctuations.

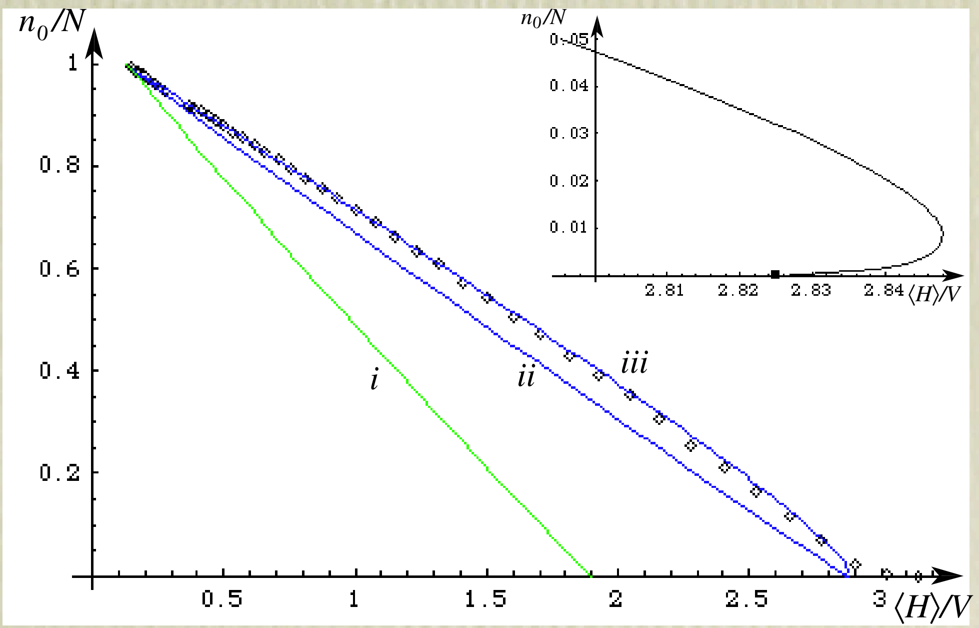


FIG. 2 (color online). Condensate fraction n_0/N vs total energy density $\langle H \rangle / V$, where $\langle H \rangle = E + E_0$, E_0 being the condensate energy [see Eq. (9)]. Points (◇) refer to numerical simulations of the NLS Eq. (1) with 64^3 modes ($N/V = 1/2$). The straight line (i) [(ii)] corresponds to the continuous Eq. (6) [discretized Eq. (7)] approximation. Curve (iii) refers to condensation in the presence of nonlinear interactions [from Eq. (9)], which makes the transition to condensation subcritical, as illustrated in the inset (with 1024^3 modes). Each point (◇) corresponds to an average over 10^3 time units.

What is an absolute equilibrium for GPE?

Grand canonical

New algorithm to generate absolute equilibrium

$$P_{\mathrm{stat}} = \frac{1}{\mathcal{Z}} e^{-\beta F} \qquad F = H - \mu N - \cancel{\mathbf{W} \cdot \mathrm{P}}$$

Non Gaussian

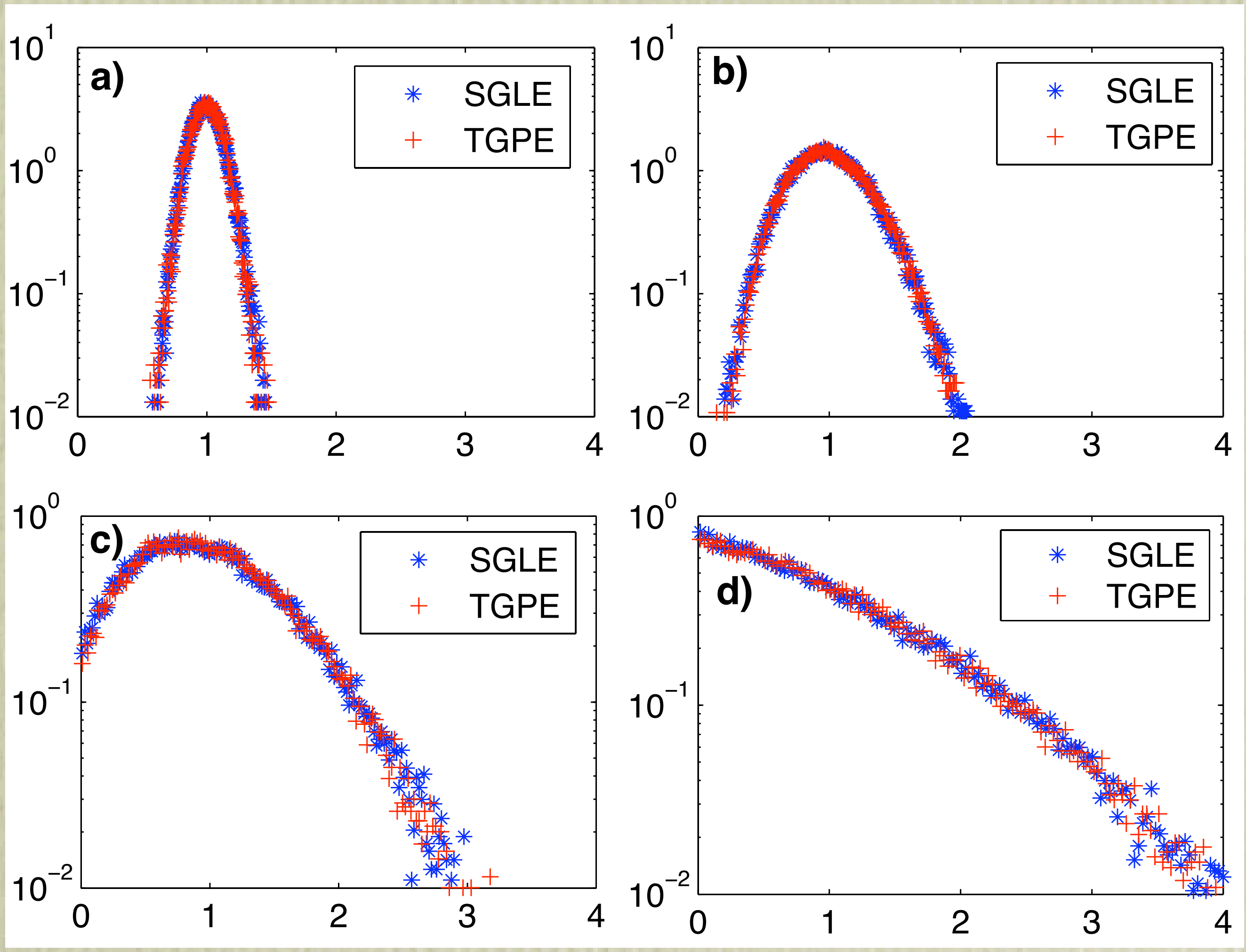
$$\hbar \frac{\partial A_{\mathbf{k}}}{\partial t} = -\frac{1}{V} \frac{\partial F}{\partial A^*_{\mathbf{k}}} + \sqrt{\frac{2\hbar}{V\beta}}\, \hat{\zeta}(\mathbf{k}, t)$$

$$\langle \zeta(\mathbf{x}, t) \zeta^*(\mathbf{x}', t') \rangle = \delta(t - t') \delta(\mathbf{x} - \mathbf{x}'),$$

$$\hbar \frac{\partial \psi}{\partial t} = \mathcal{P}_{\mathrm{G}}[\frac{\hbar^2}{2m} \nabla^2 \psi + \mu \psi - g \mathcal{P}_{\mathrm{G}}[|\psi|^2]\psi - \cancel{i\hbar \mathbf{W} \cdot \nabla \psi}] + \sqrt{\frac{2\hbar}{V\beta}} \mathcal{P}_{\mathrm{G}}[\zeta(\mathbf{x}, t)]$$

Partition function can be analytically obtained at low temperatures

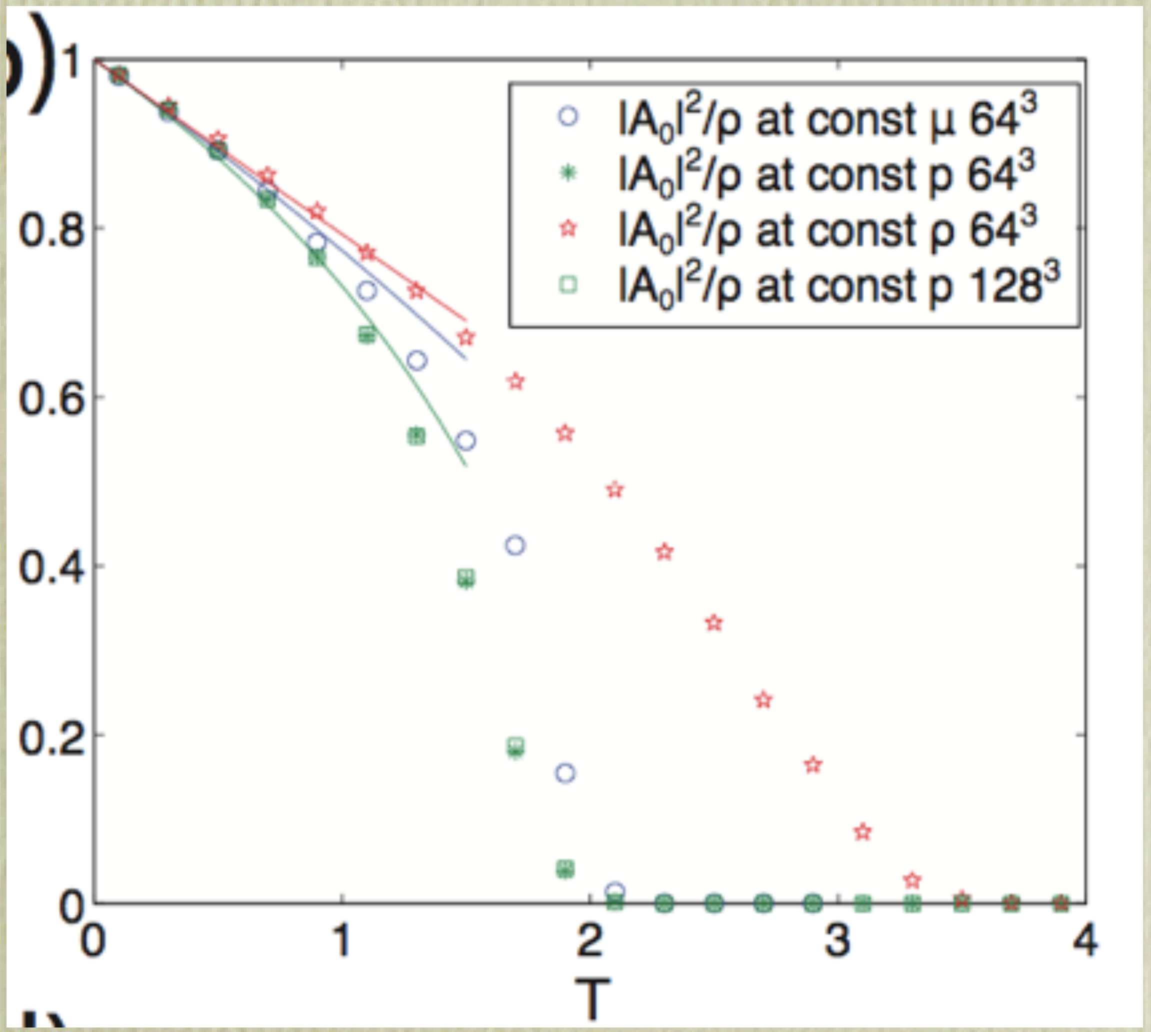
Micro canonical versus grand canonical

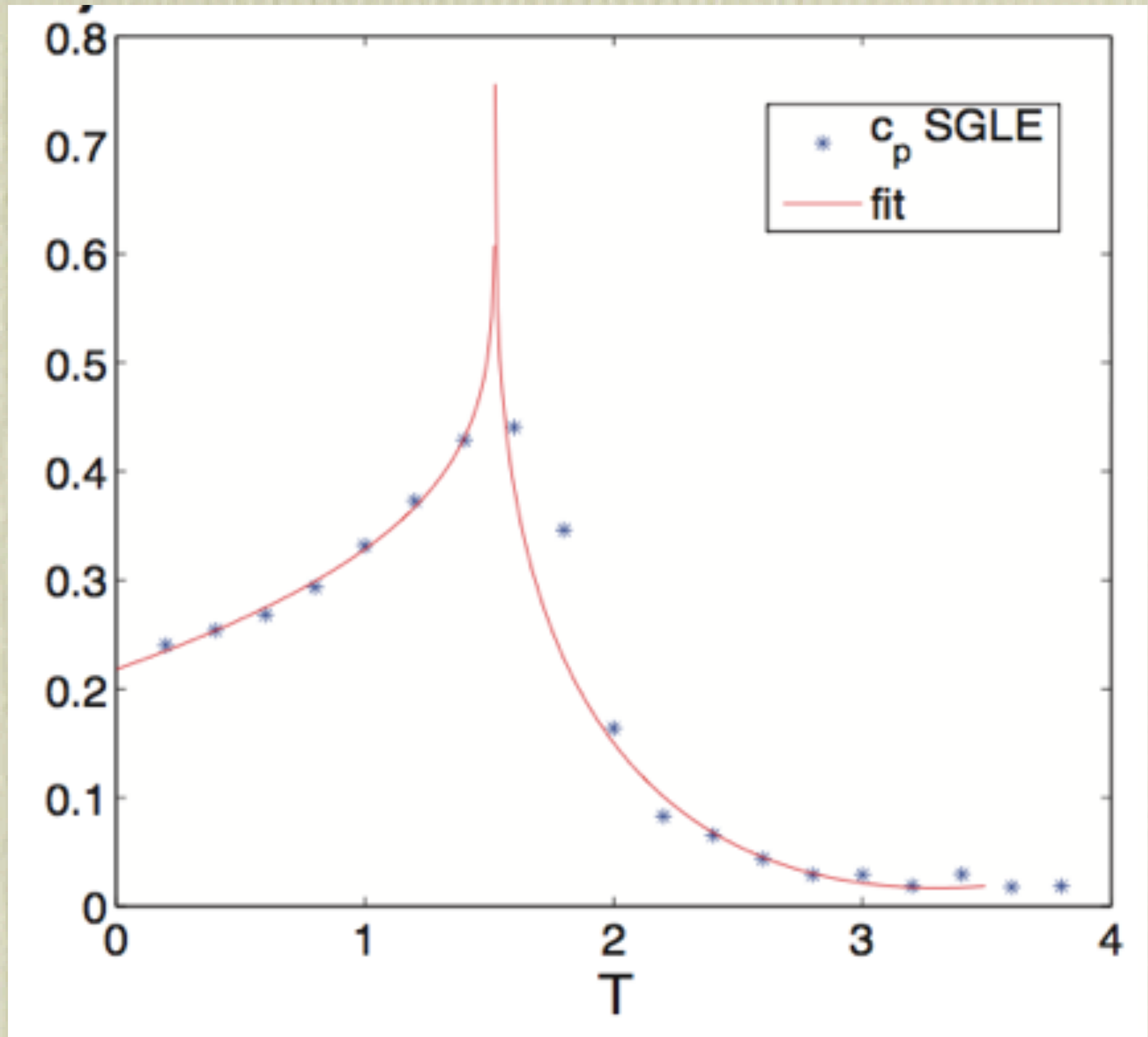


Density histograms

H	T	TGPE time steps	SGLE time steps
0.09	0.09	40000	9600
0.5	0.5	20000	9600
1.96	1.8	20000	9600
4.68	4	20000	5000

Condensation transition



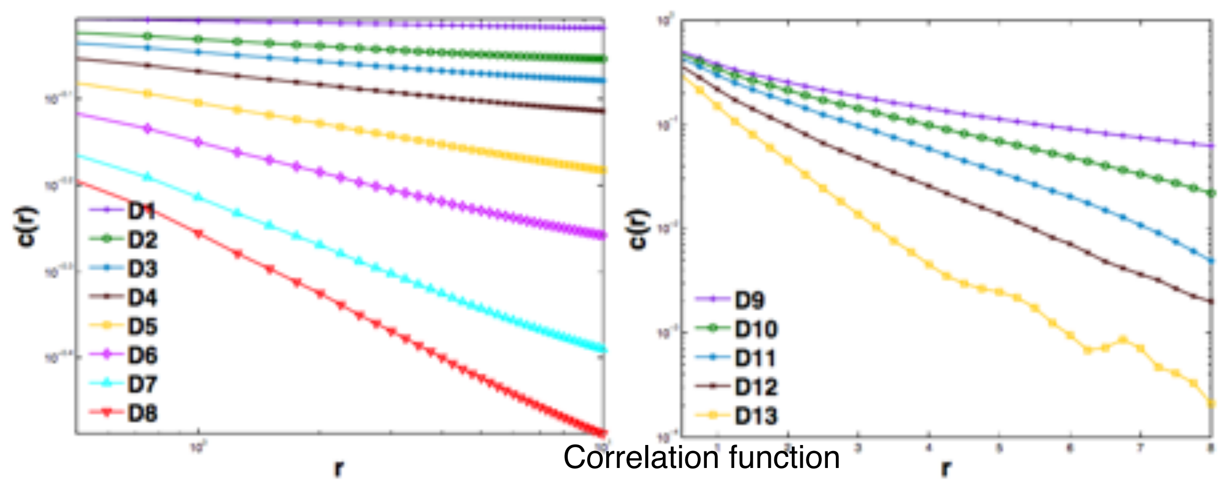


$$\lambda - \phi^4 \quad (D = 3, n = 2)$$

2D BKT transition

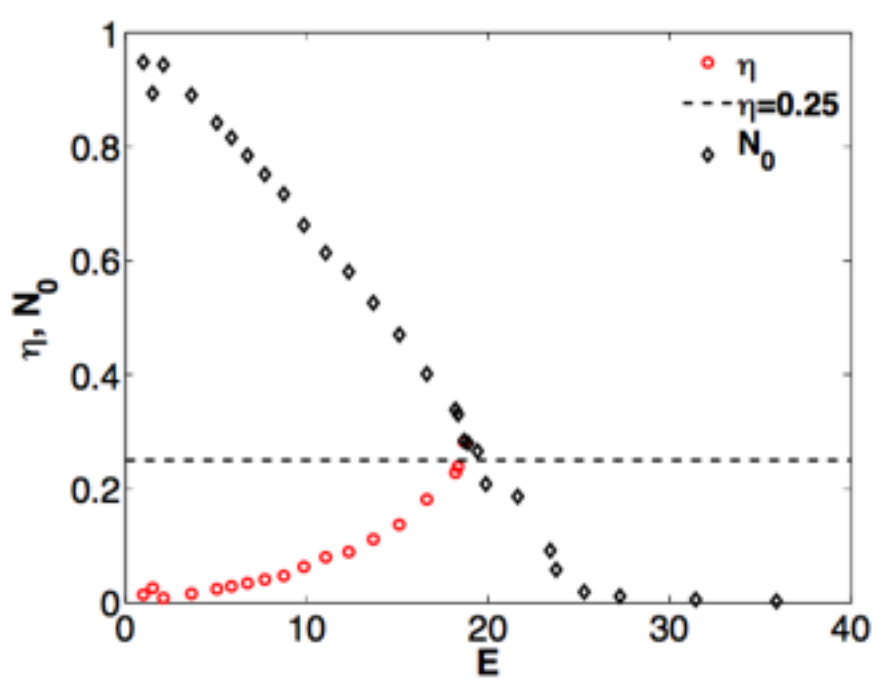
Vishwanath Shukla, Marc Brachet and Rahul Pandit
Turbulence in the two-dimensional Fourier-truncated Gross–Pitaevskii equation
New J. Phys. 15 113025 (2013)

Below transition



Above transition

[Left] Log-log plot of $c(r)$ vs. r ($E < E_{\mathrm{BKT}}$, $N_c = 128$); [Right] Semilog-y plot $c(r)$ vs. r ($E > E_{\mathrm{BKT}}$, $N_c = 128$).

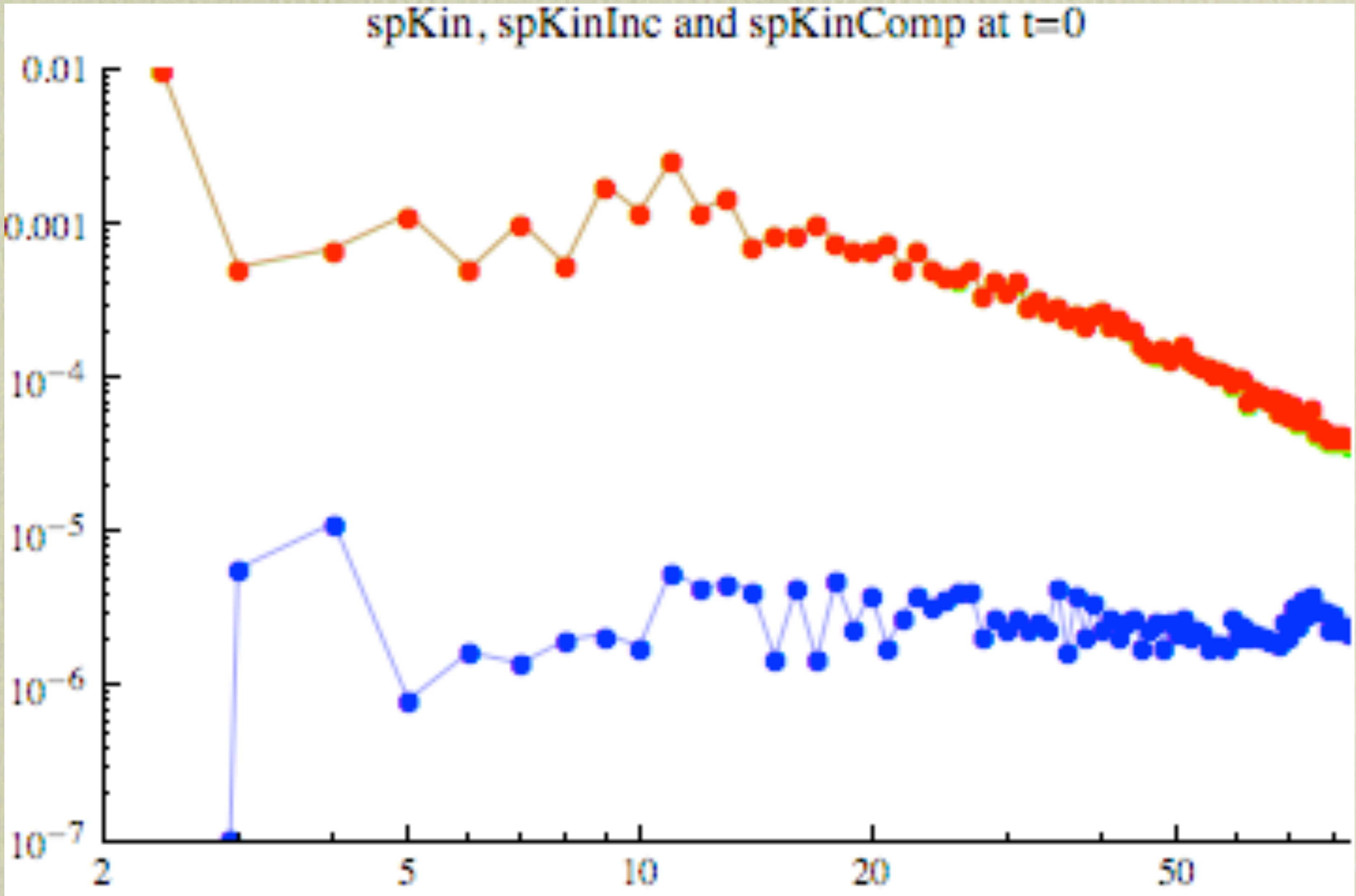


Condensate fraction
and
Power exponent

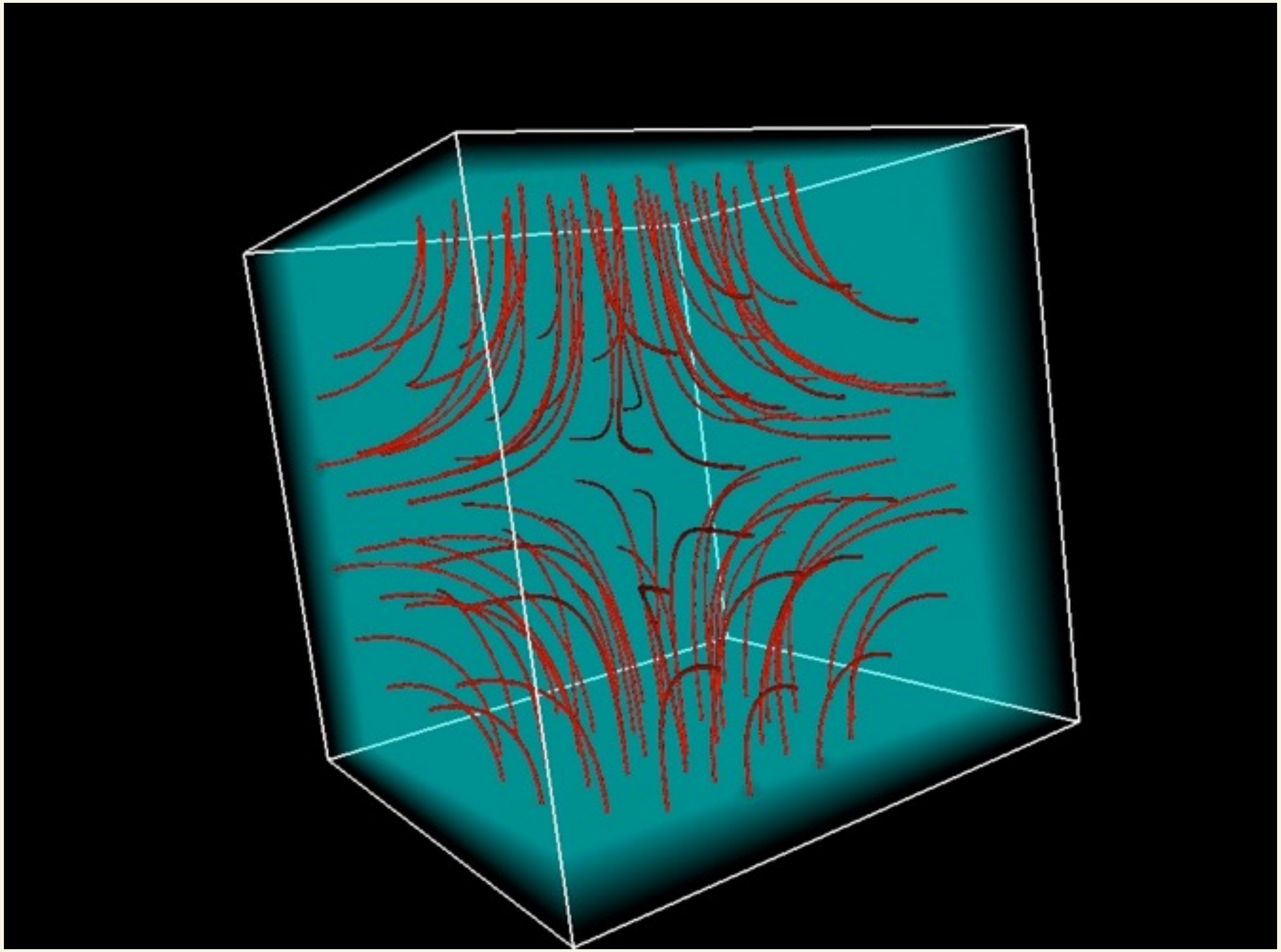
Dynamics of thermalization in the GPE

Kinetic energy spectrum

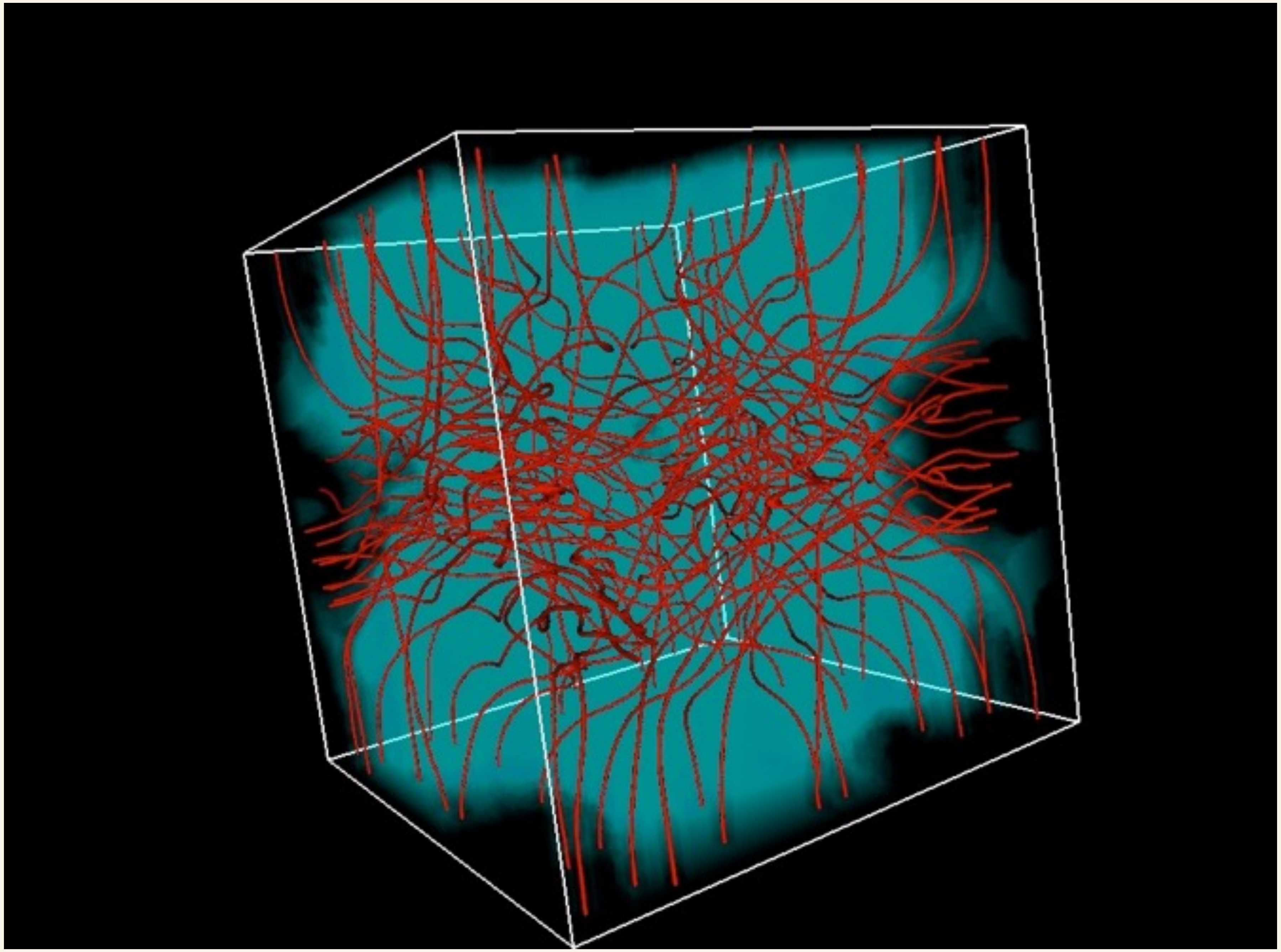
- $E_{kin\ tot}(k)$
- $E_{kin\ inc}(k)$
- $E_{kin\ comp}(k)$



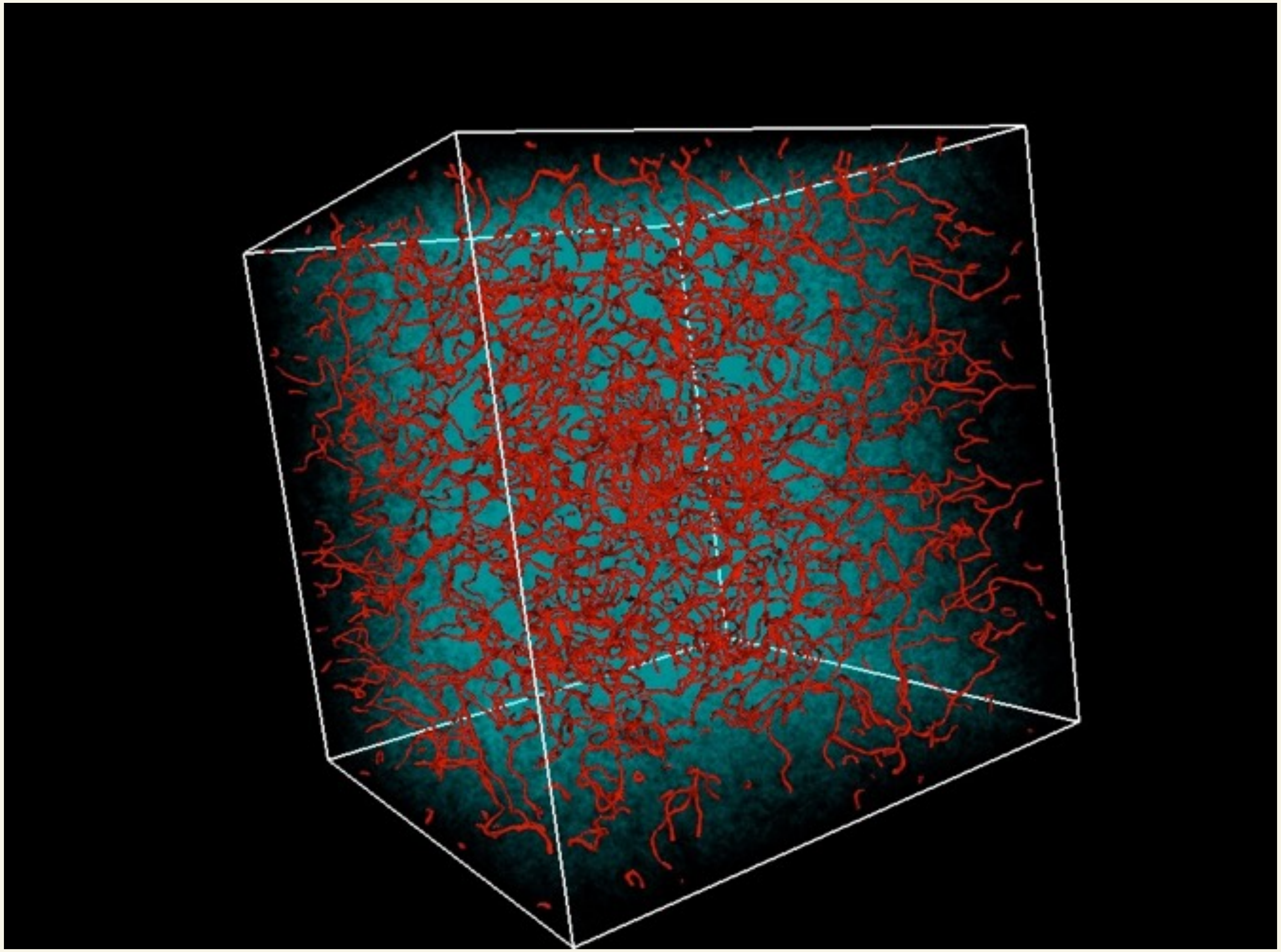
Taylor-Green vortex



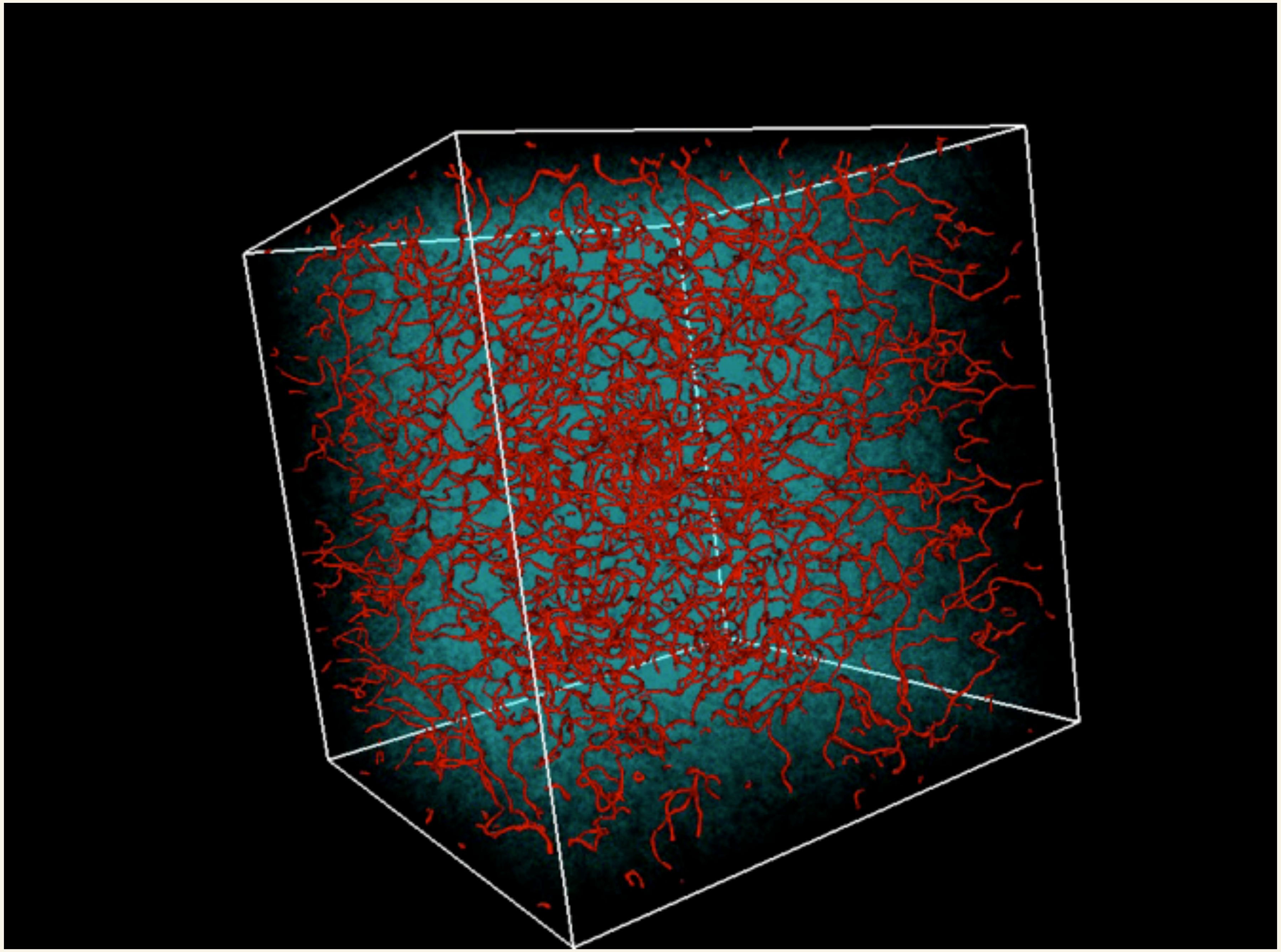
Taylor-Green vortex



Taylor-Green vortex



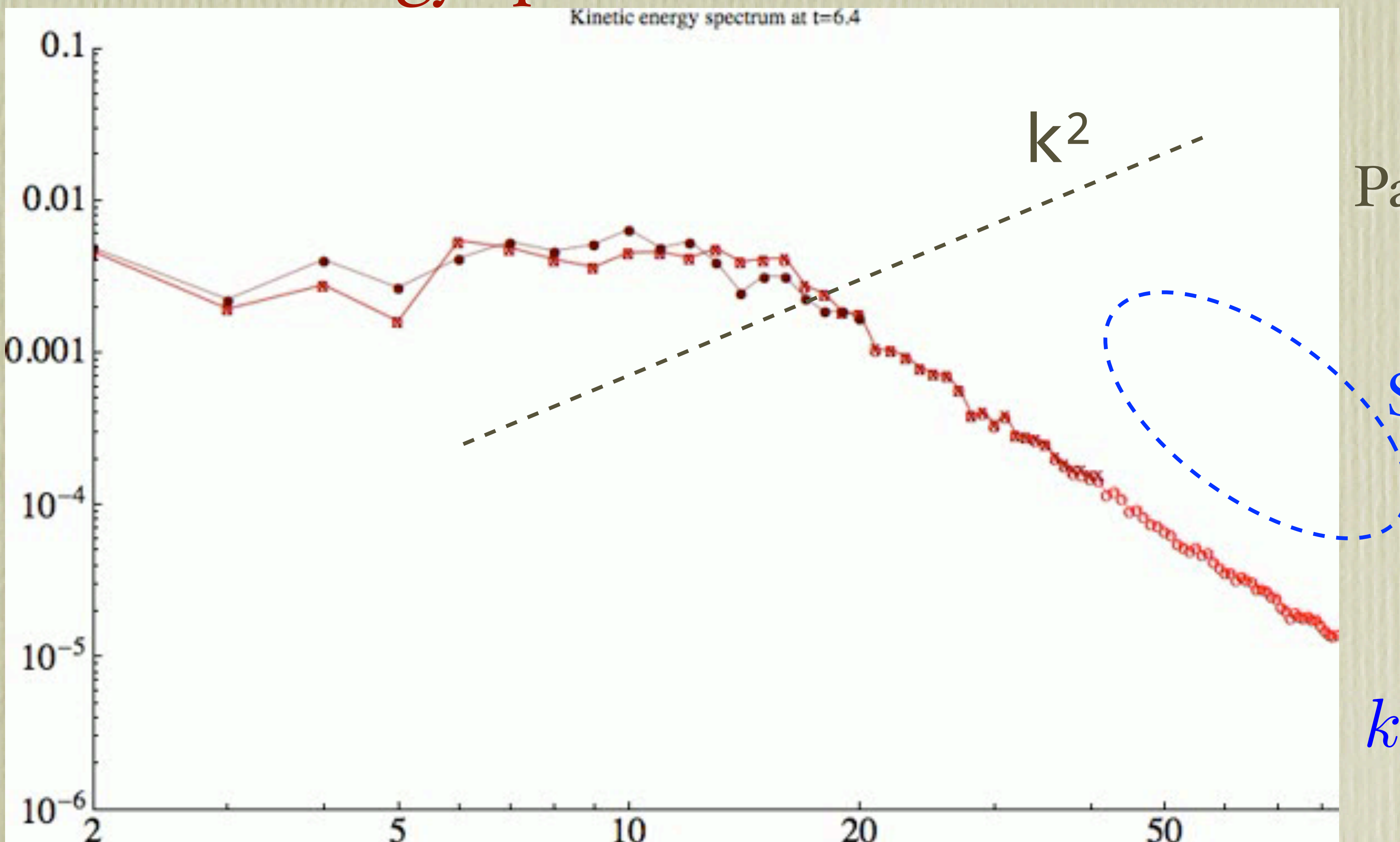
Taylor-Green vortex



Dispersive "bottleneck" for thermalization of waves

Variable $\xi k_{\max}$ (ξ fixed, different resolutions)

Kinetic energy spectrum



Partial thermalization starts

Spectral convergence

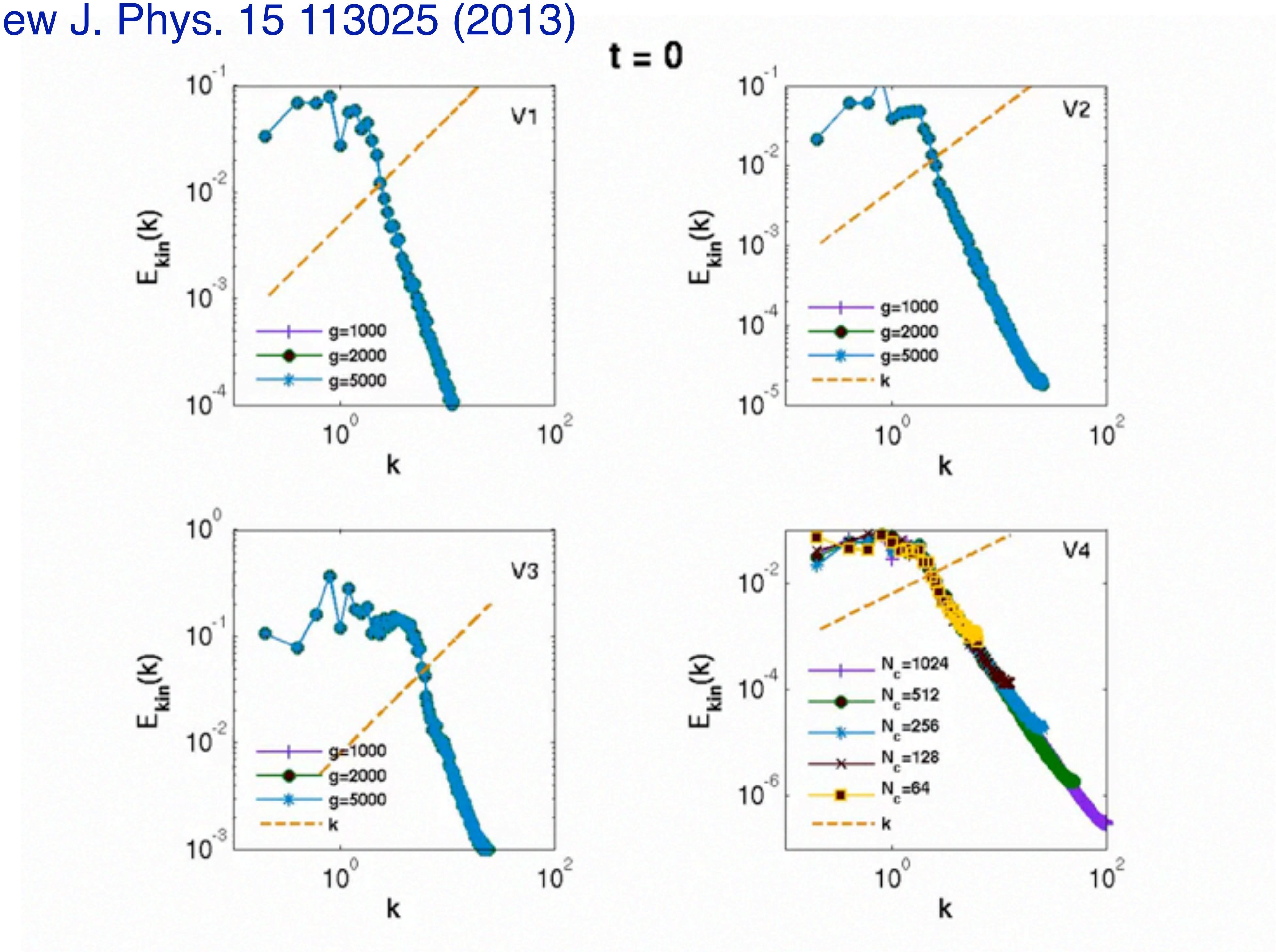
$k_{\max}$-independent bottleneck

Self truncation in 2D

Vishwanath Shukla, Marc Brachet and Rahul Pandit

Turbulence in the two-dimensional Fourier-truncated Gross–Pitaevskii equation

New J. Phys. 15 113025 (2013)



[Top Left] k0 = 5Δk and σ = 2Δk, [Top Right] k0 = 15Δk and σ = 2Δk, [Bottom Left] k0 = 35Δk and σ = 5Δk, and [Bottom Right] different Nc.

Is it possible to obtain Self-truncation in the framework of classical fluids?

Euler-Voigt-α model

$$[1+(-\alpha^2\nabla^2)^{\frac{\beta}{2}}]\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p,$$

$$\nabla\cdot\mathbf{u} = 0,$$

This dynamics conserves the generalized Energy with spectrum:

$$E_\alpha(k,t) = \frac{1}{2}\sum_{\substack{\mathbf{k}\in\mathbb{Z}^3 \\ k-1/2<|\mathbf{k}|<k+1/2}} [1+(\alpha k)^\beta]|\widehat{\mathbf{u}}(\mathbf{k},t)|^2.$$

Let us remark that the differential operator multiplying the right-hand side of our generalized 3D Euler-Voigt-α model Eq. (2) can be written in Fourier space as $1+(\alpha k)^\beta = 1 + (k/k_\alpha)^\beta$. The formal limit $\beta \to \infty$ of Eq. (2) thus corresponds to a standard spherical Galerkin truncation [$\hat{\mathbf{u}}(\mathbf{k}) = 0$ for $|\mathbf{k}| > k_{\max}$] of the Euler Eq. (3) at $k_{\max} = k_\alpha$. Note that a somewhat

Numerical method

We consider here solutions of Eq. (2) that correspond to the so-called Taylor-Green (TG) [24] (2π-periodic) initial data $\mathbf{u}(x,y,z,0) = \mathbf{u}^{\mathrm{TG}}(x,y,z)$, with

$$\mathbf{u}^{\mathrm{TG}} = [\sin(x)\cos(y)\cos(z), -\cos(x)\sin(y)\cos(z), 0]. \quad (9)$$

The simulations reported in this paper were performed using a special purpose symmetric parallel code developed from that described in Refs. [23,25–27]. The code uses the symmetries of the Taylor-Green initial data to speed-up computations and optimize memory usage. The workload for a time step is (roughly) twice that of a general periodic code running at a quarter of the resolution. Specifically, at a given computational cost, the ratio of the largest to the smallest scale available to a computation with enforced Taylor-Green symmetries is enhanced by a factor of 4 in linear resolution. This leads to a factor of 32 savings in total computational time and memory usage. The code is based on FFTW and a hybrid MPI-OpenMP scheme derived from that described in Ref. [28]. At resolution 2048^3 we used 512 MPI processes, each process spawning 8 OpenMP threads.

Results

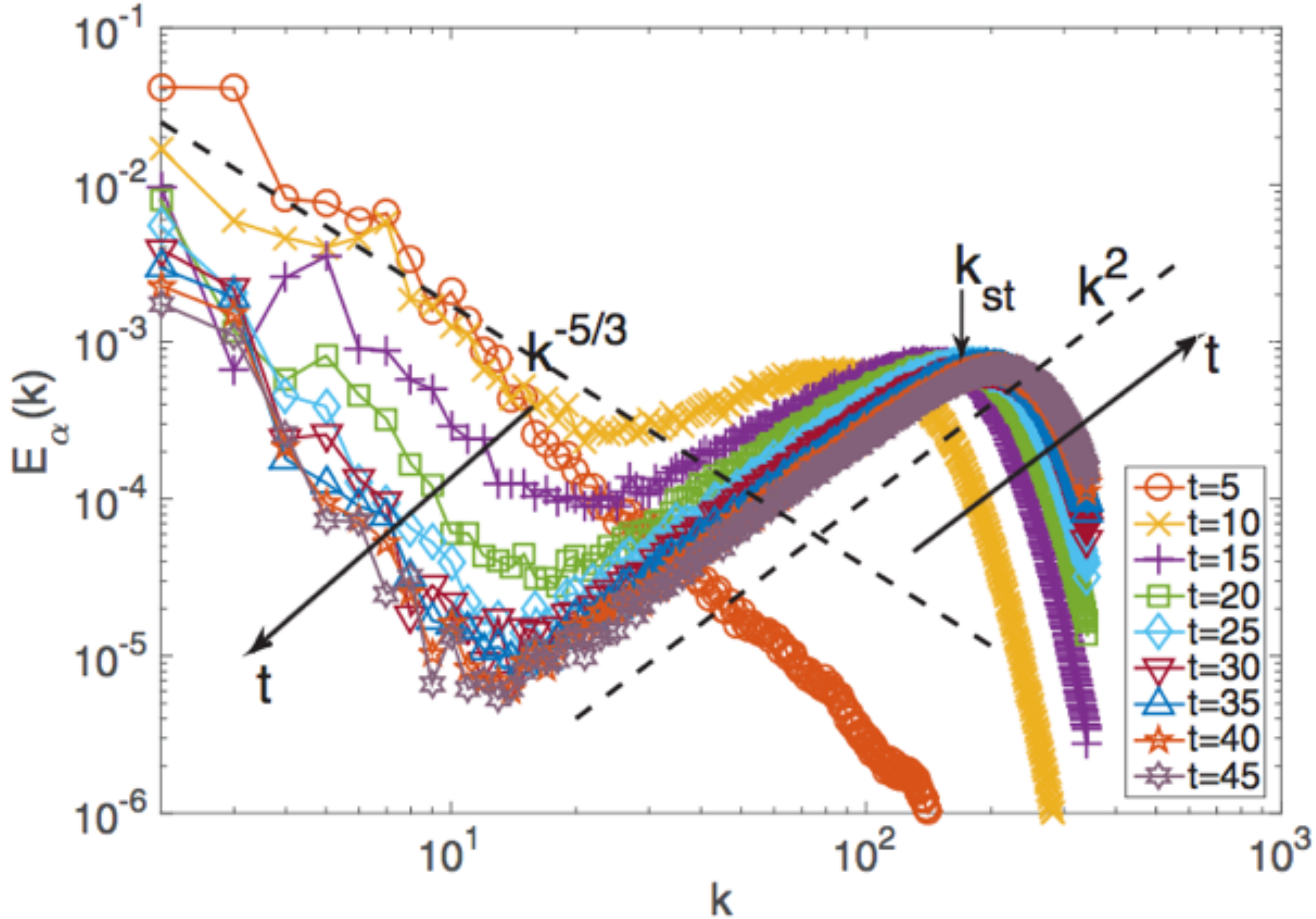
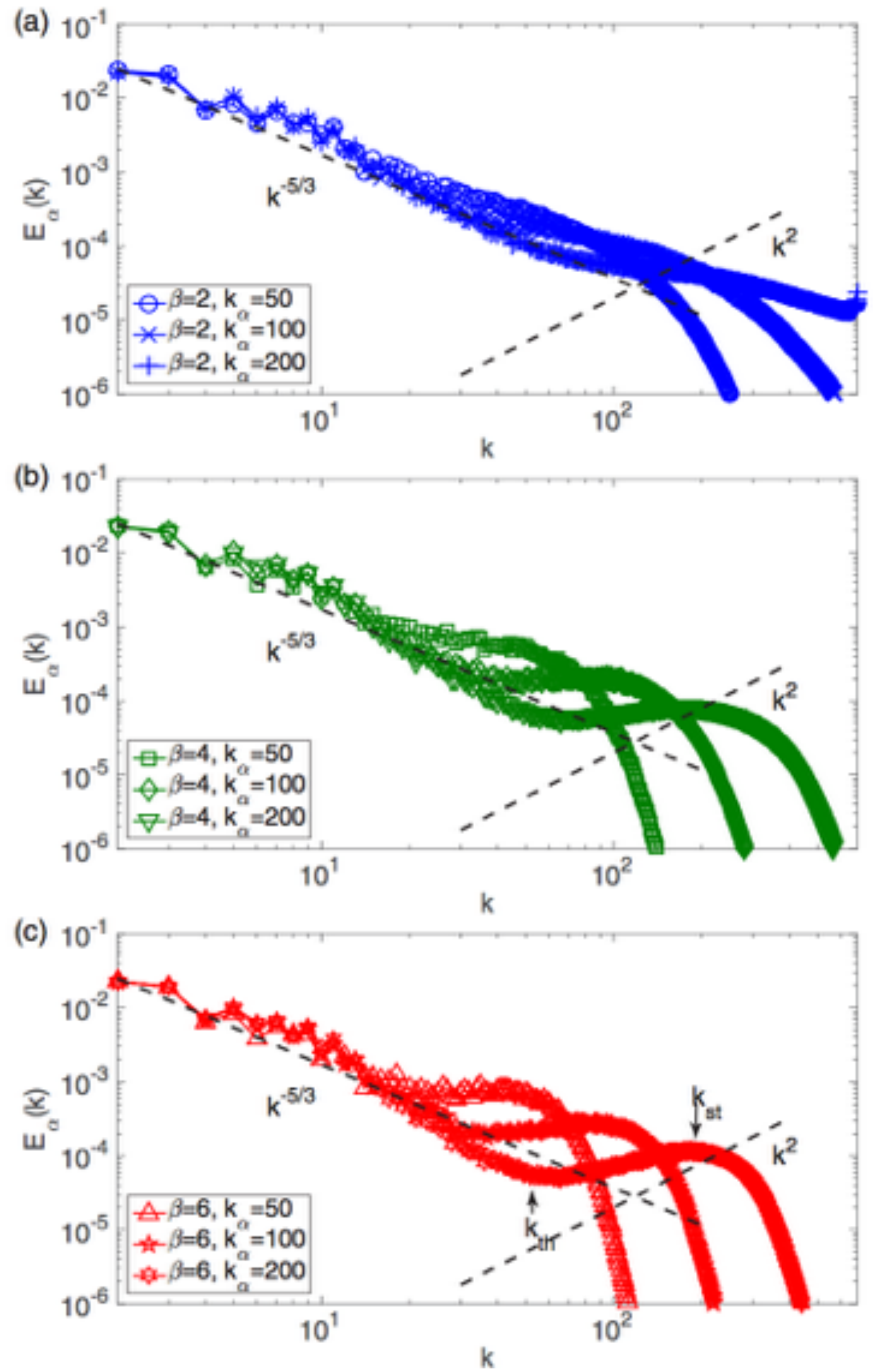


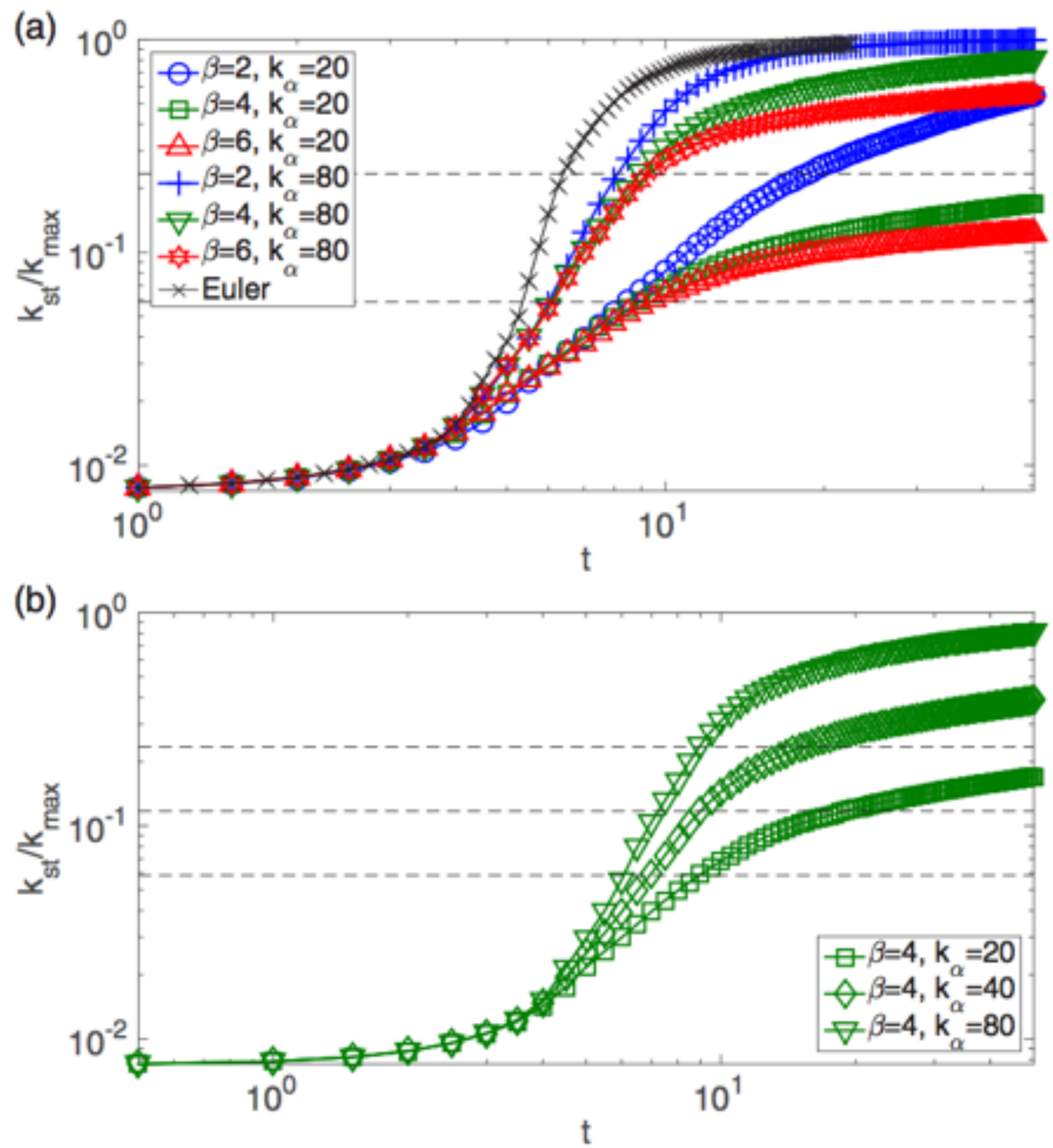
FIG. 1. (Color online) Temporal evolution (indicated by arrows) of the energy spectrum $E_\alpha(k)$ for $\beta = 4$ and $k_\alpha = 80$. Resolution 1024^3 ($k_{\max} = 342$). The dashed lines respectively display the Kolmogorov $k^{-5/3}$ and the equipartition k^2 scaling. The self-truncation wavenumber is indicated by the small vertical arrow.

Changing ß



(a)
(b)
(c)
$k^{-5/3}$
k^2
$E_\alpha(k)$
k
β=2, k_α=50
β=2, k_α=100
β=2, k_α=200
β=4, k_α=50
β=4, k_α=100
β=4, k_α=200
β=6, k_α=50
β=6, k_α=100
β=6, k_α=200
k_{st}
k_{th}

Evolution of self-truncation wavenumber



Self-similar long time behavior

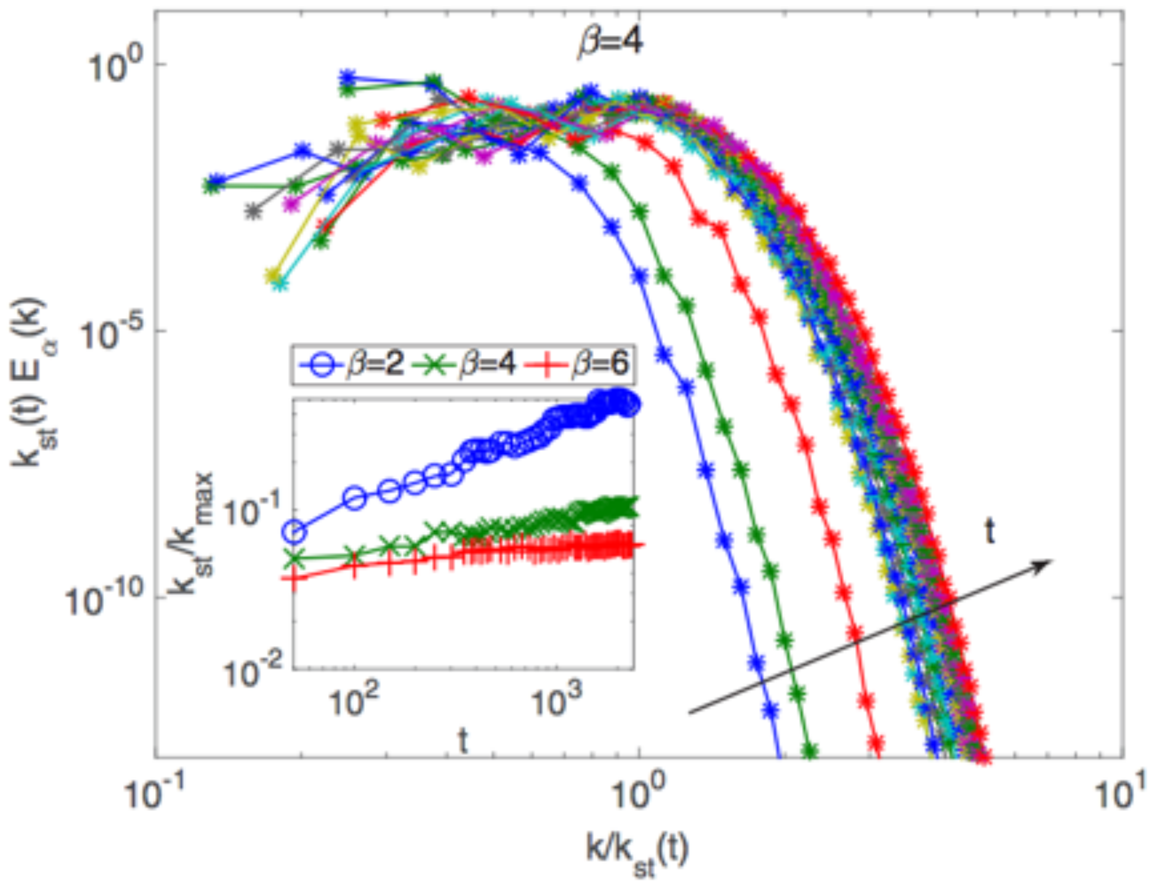
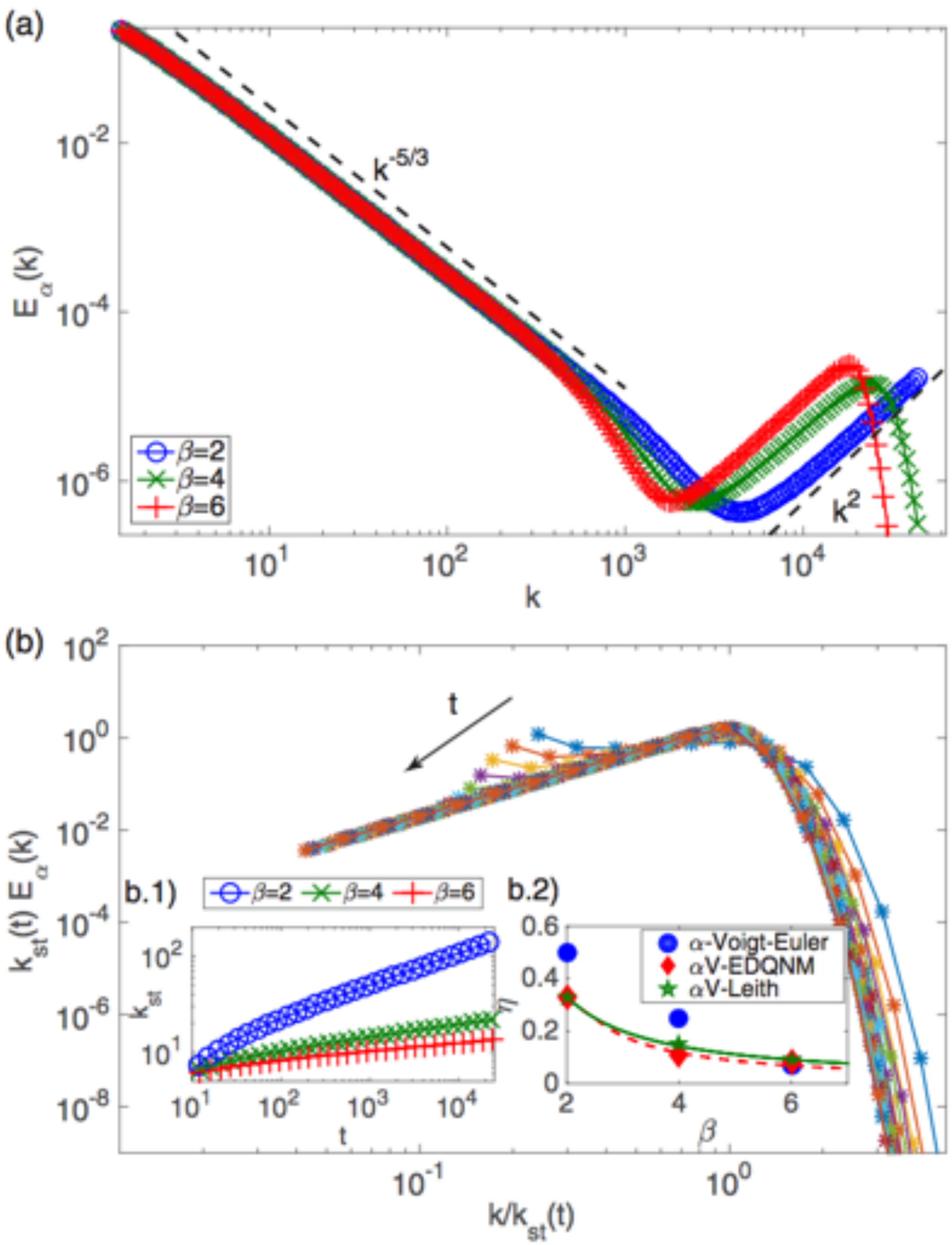


FIG. 6. (Color online) Temporal evolution of the self-similar function $\Psi[k/k_{\rm st}(t)] = E_\alpha(k,t)k_{\rm st}(t)$ [see Eq. (13)] for $\beta = 4$ and $k_\alpha = 4$. Data from direct numerical resolution of Eq. (2) at resolution 512^3. The inset shows the temporal evolution of $k_{\rm st}(t)/k_{\rm max}$ for

$$E_\alpha(k,t) = \frac{E_0}{k_{\rm st}(t)}\Psi\left[\frac{k}{k_{\rm st}(t)}\right],$$

EDQNM models



$$\frac{\partial E(k,t)}{\partial t} = \frac{1}{1+\alpha^\beta k^\beta} T_{NL}(k,t),$$

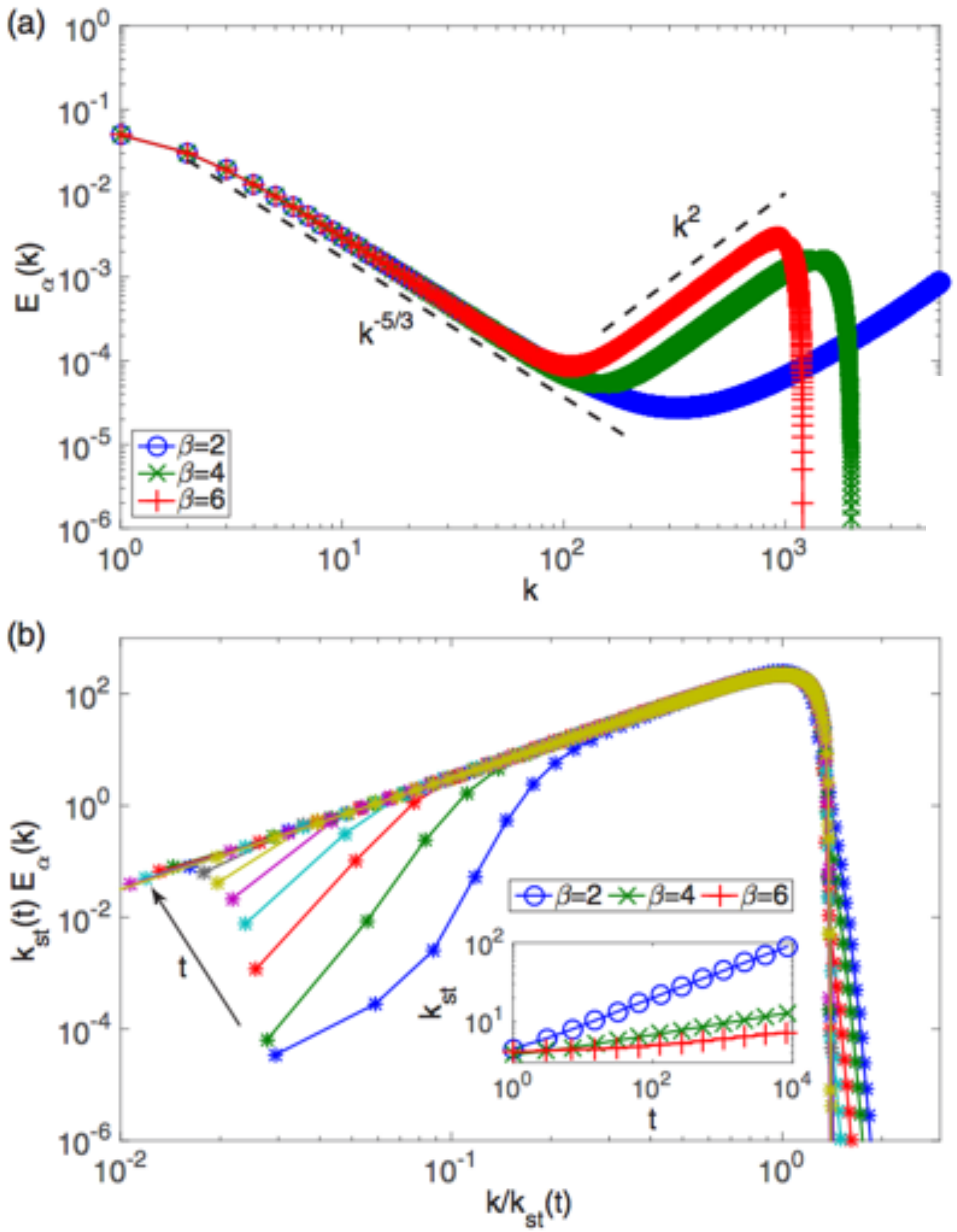
where the nonlinear transfer T_{NL} is modeled as

$$T_{\mathrm{NL}}(k,t) = \iint_\Delta \Theta_{kpq}(xy+z^3)\left[\frac{k^2 p E(p,t)E(q,t)}{1+\alpha^\beta k^\beta} - \frac{p^3 E(q,t)E(k,t)}{1+\alpha^\beta p^\beta}\right]\frac{dp\,dq}{pq}.$$

$$\Theta_{kpq} = \frac{1-\exp[-(\eta_k+\eta_p+\eta_q)t]}{\eta_k+\eta_p+\eta_q}.$$

$$\eta_k = \lambda'\sqrt{\int_0^k \frac{s^2 E(s,t)}{1+\alpha^\beta s^\beta}\,ds}.$$

Leith model



$$\frac{\partial E_\alpha}{\partial t} = \frac{2q\gamma}{3}\frac{\partial}{\partial k}\left[\tau_3^\alpha(k)k^7\, E_\alpha \frac{\partial}{\partial k}(k^{-2}E_\alpha)\right],$$

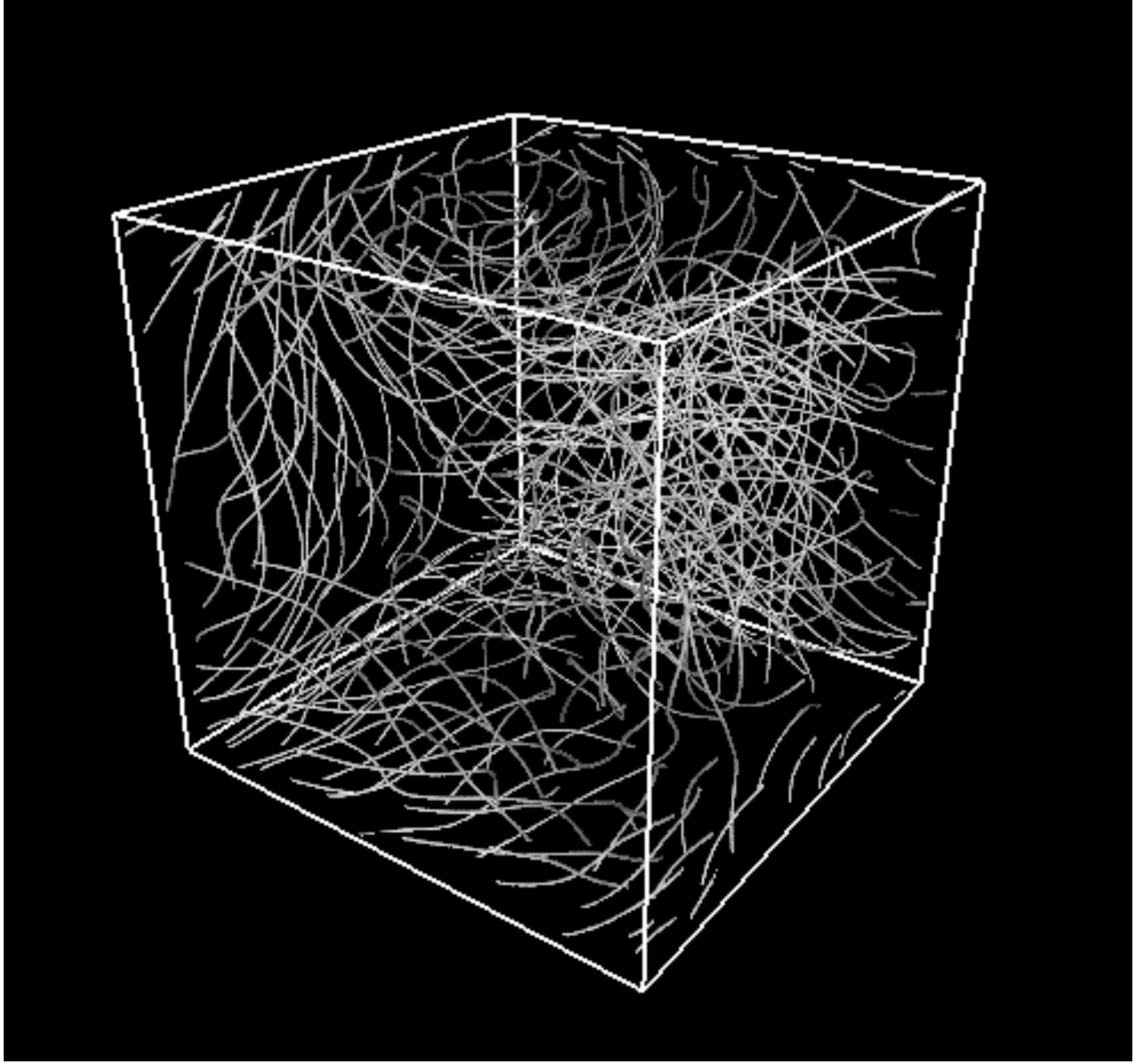
Results

TABLE II. Values of the exponent η of the self-truncation wave number $k_{st}(t) \sim t^{\eta}$ [see Eq. (12)] obtained from direct numerical simulation of the Euler-Voigt-α model Eq. (2), αV-EDQNM [Eqs. (14)–(17)], and αV-Leith model ($r = 2$) [Eq. (24)].

	$\beta = 2$	$\beta = 4$	$\beta = 6$
αV-Euler	$0.5 \pm 6 \times 10^{-3}$	$0.25 \pm 3 \times 10^{-3}$	$0.07 \pm 5 \times 10^{-3}$
αV-EDQNM	$0.33 \pm 5 \times 10^{-5}$	$0.11 \pm 9 \times 10^{-5}$	$0.085 \pm 1 \times 10^{-4}$
αV-Leith	0.33 ± 10^{-6}	$0.15 \pm 2 \times 10^{-4}$	$0.09 \pm 9 \times 10^{-6}$

Conclusion

- Self-truncation obtained for classical fluid
- Is there a universal exponent?
- What experimental system?



Thank you!