Spectral properties of superfluid turbulence

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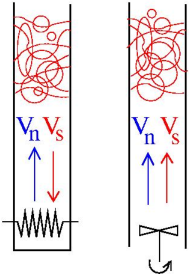
Outline

- Energy spectrum
- Vortex line density spectrum
- New numerical results shed light into these spectra

Superfluid turbulence

^{4}He turbulence, statistical steady state, away from boundaries, created either thermally (counterflow) or mechanically (coflow), $T > 1$ K regime

Measure vortex line density L (temperature gradients, second sound, etc), infer typical intervortex distance $\ell \sim L^{-1/2}$



- Thermally: $L \sim V_{ns}^2$ (Vinen 1956)
  $V_{ns} = V_n - V_s \sim \dot{Q}$
  $V_{ns} =$ counterflow
  $\dot{Q} =$ applied heat flux
- Mechanically: $\ell/D \sim Re^{-3/4}$ (Salort *et al* 2011)
  hence $L \sim Re^{3/2}$
  Reynolds $Re = V_n D/\nu_n \sim (D/\eta)^{4/3}$
  $\eta =$ dissipation scale, D system scale
  $\nu_n =$ kinematic viscosity
  $\eta =$ Kolmogorov length

Why spectra ?

Spectra help to understand the property of turbulence

- Energy spectrum:
  tells about distribution of energy over length scales

$$E = \frac{1}{V}\int_V \frac{\mathbf{v}^2}{2} dV = \int_0^\infty E(k)dk$$

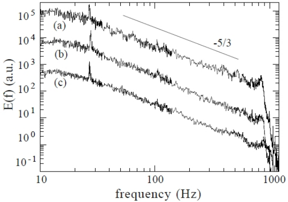
$$E(k) \sim k^{-5/3} \quad \text{in } 1/D \ll k \ll 1/\eta$$

Numerics: measure $\mathbf{v}$ vs $\mathbf{r}$ at given t
Experiments: measure $\mathbf{v}$ vs t at given $\mathbf{r}$

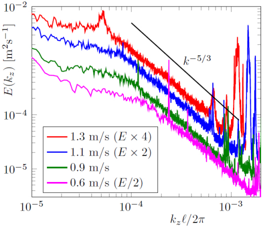
- Vortex line density spectrum:
  should tell about 'worms'
  (Sasa *et al* 2011, Baggaley 2012)

Experimental spectra

- Energy spectrum:

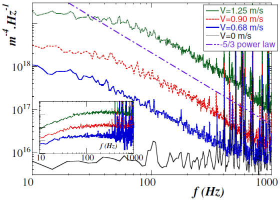


(Maurer & Tabeling 1998)



(Roche *et al* 2007)

- Vortex line density spectrum:



(Roche *et al* 2007)

Theoretical spectra

- **Energy spectrum:** $E(k) \sim k^{-5/3}$

Nore, Abid & Brachet 1997 ($T = 0$, GPE)
Kobayashi & Tsubota 2005 ($T = 0$, GPE)
Araki, Tsubota, Nemirovskii 2002 ($T = 0$, Biot-Savart)
Baggaley, CFB, Shukurov & Sergeev 2012 ($T = 0$, Biot-Savart)
Baggaley & CFB 2011 ($T \neq 0$, Biot-Savart)
L'vov, Nazarenko & Skrbek 2006 ($T \neq 0$, Leith model)
Roche, CFB & Leveque 2009 ($T \neq 0$, HVBK model)
Wacks & CFB 2011 ($T \neq 0$, shell model)
etc

- **Vortex line density spectrum:**
Counterflow:
Nemirovskii 2012,
CFB, Swanson & Donnelly 1982

Numerical method

Vortex lines as spaces curves $\mathbf{s}(\xi, t)$ in periodic box

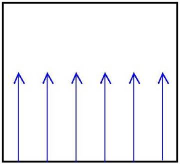
$$\frac{d\mathbf{s}}{dt} = \mathbf{v}_s^{ext} + \mathbf{v}_s^{self} + \alpha \mathbf{s}' \times (\mathbf{v}_n^{ext} - \mathbf{v}_s^{ext} - \mathbf{v}_s^{self}) - \alpha' \mathbf{s}' \times (\mathbf{s}' \times (\mathbf{v}_n^{ext} - \mathbf{v}_s^{ext} - \mathbf{v}_s^{self}))$$

$$\mathbf{v}_s^{self}(\mathbf{s}) = \frac{\kappa}{4\pi} \oint \frac{(\mathbf{r} - \mathbf{s}) \times \mathbf{dr}}{|\mathbf{r} - \mathbf{s}|^3} \qquad (\mathrm{Biot-Savart})$$

Desingularization and reconnections

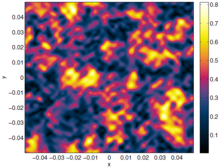
Prescribed $\mathbf{v}_n^{ext}$ and $\mathbf{v}_s^{ext}$

Models for $\mathbf{v}_n$



- Uniform normal flow in a given direction

  $\mathbf{v}_n^{ext} = V_n \hat{\mathbf{x}}, \quad \mathbf{v}_s^{ext} = \mathbf{0}$

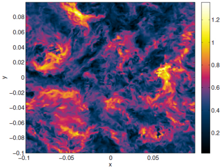


- Synthetic turbulence

$$\mathbf{v}_n^{ext} = \sum_{m=1}^{m=M} (\mathbf{A}_m \times \mathbf{k}_m \cos\phi_m + \mathbf{B}_m \times \mathbf{k}_m \sin\phi_m),$$

$$\phi_m = \mathbf{k}_m \cdot \mathbf{s} + f_m t, \quad f_m = \sqrt{k_m^3 E(k)},$$

$$E(k_m) \sim k_m^{-5/3}, \quad Re = (k_M / k_1)^{4/3}$$

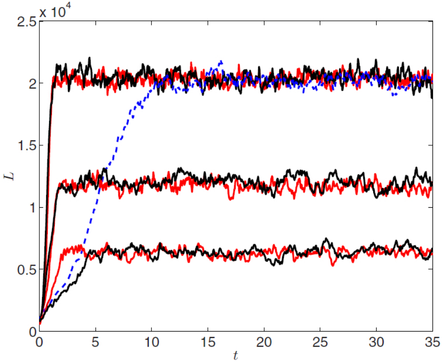


- Turbulent solution $\mathbf{v}_n^{ext}$ of Navier-Stokes (frozen)

  1024^3, $Re \approx 3200$

  (John Hopkins Turbulence Database)

Saturated vortex tangles

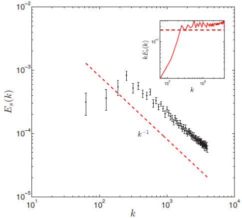


Measurements done in the saturated regime

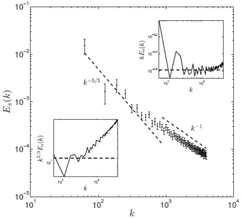
Red: uniform normal fluid
Black: synthetic turbulence
Blue: Navier-Stokes turbulence

Energy spectra: bump vs Kolmogorov

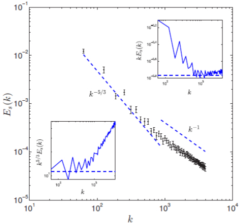
- Superfluid energy spectrum:



uniform normal fluid bump, lacks energy at small k



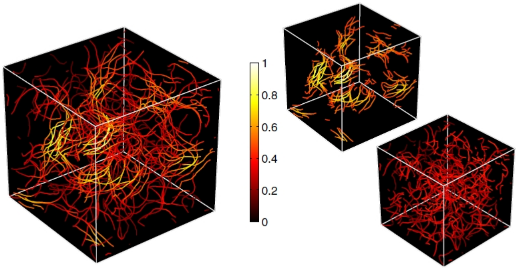
synthetic turbulence Kolmogorov, most energy at small k

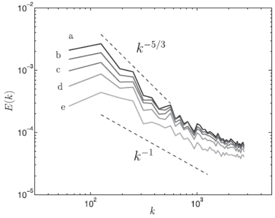


Navier-Stokes turbulence (Kolmogorov)

Origin of Kolmogorov spectrum

Kolmogorov law arises from polarized vortex bundles

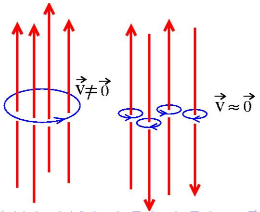




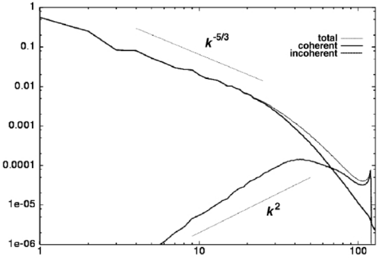
tot. & pol. $E(k) \sim k^{-5/3}$
unpol. $E(k) \sim k^{-1}$

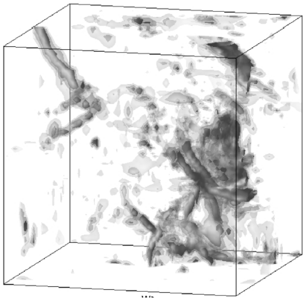
Colour-coded by coarse-grained vorticity
Left: total lines
Right: polarized (yellow), unpolarized (red)

Baggaley, Laurie, CFB 2012



Vortex tubes in classical turbulence





Wavelet decomposition (Farge *et al* 2001) shows that the $k^{-5/3}$ spectrum of classical turbulence arises from vortex tubes

Bumpy vs Kolmogorov spectrum

Classical fluid:

$$\text{Cascade exists if } \tau_d = \frac{r^2}{\nu} \gg \tau_r = \frac{r}{u_r} \quad \Rightarrow \quad Re_r = \frac{u_r r}{\nu} \gg 1$$

Helium:

$$\frac{dV_s}{dt} = \cdots - \alpha\kappa L(V_s - V_n) = \cdots - \frac{V_s}{\tau_f}$$

$$\tau_f = \frac{1}{\alpha\kappa L\beta} \quad \text{where } \beta = \rho/\rho_n \text{ (counterflow) and } \quad \beta \ll 1 \text{ (coflow)}$$

Cascade exists only if $\tau_f \gg \tau_r$ and $r \gg \ell$
hence Kolmogorv scaling is expected only for

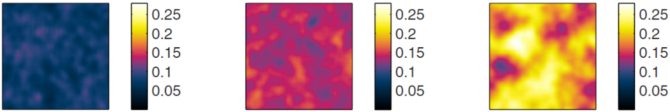
$$\ell \ll r \ll r_c = \frac{\epsilon^{1/2}\ell^3}{(\alpha\kappa\beta)^{3/2}}$$

limiting the Kolmogorov scaling at large scales for counterflow but not for coflow.

Spatial structure

- Vortex lines:

- Superfluid energy density smoothed over intervortex spacing:

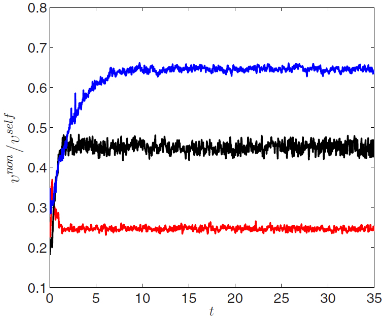


Left: uniform normal fluid,
Middle: synthetic turbulence,
Right: Navier-Stokes turbulence

Local vs nonlocal contribution

$$\mathbf{v}_s^{self}(\mathbf{s}_j) = \mathbf{v}_s^{loc}(\mathbf{s}_j) + \mathbf{v}_s^{non}(\mathbf{s}_j)$$

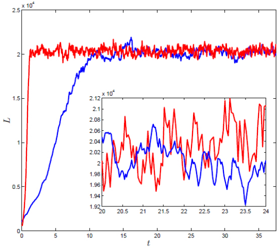
$$\mathbf{v}_s^{self}(\mathbf{s}_j) = \frac{\kappa}{4\pi} \ln \left(\frac{\sqrt{\Delta\xi_j \Delta\xi_{j+1}}}{a_0} \right) \mathbf{s}_j' \times \mathbf{s}_j'' + \frac{\kappa}{4\pi} \oint' \frac{(\mathbf{r} - \mathbf{s}_j) \times \mathbf{dr}}{|\mathbf{r} - \mathbf{s}_j|^3}$$



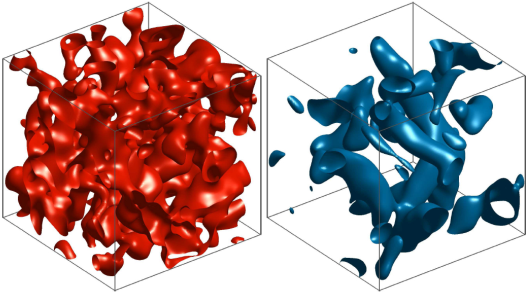
Relative nonlocal contribution

red: uniform normal fluid
black: synthetic turbulence
blue: Navier-Stokes turbulence

Spectrum of the vortex line density



Time series of $L(t)$

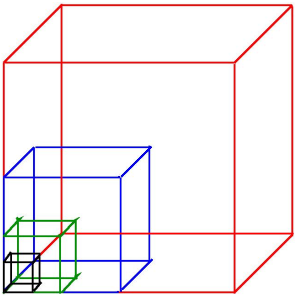


Regions of more intense L

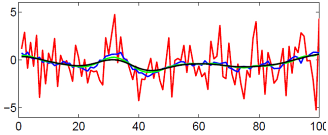
Red: uniform normal fluid; Blue: Navier-Stokes turbulence

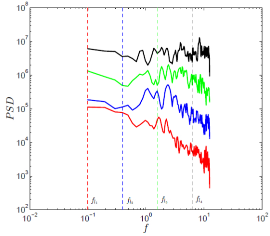
From white to coloured noise

Check effect of averaging over D^3, $(D/2)^3$, $(D/4)^3$ or $(D/8)^3$

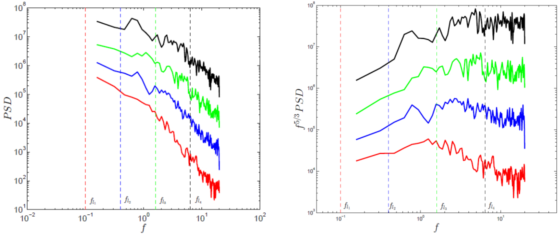


The integral of white noise is brown (f^{-2}) noise





uniform
normal fluid



Navier-Stokes turbulence
(right: compensated by $f^{5/3}$)

red: D^3
blue: $(D/2)^3$
green: $(D/4)^3$
black: $(D/8)^3$

Baggaley & CFB (2011):
spectrum $\sim f^{-5/3}$ as material
lines transported by turbulence

More topics of investigations

- **Towards low temperature**

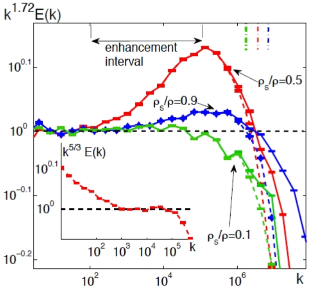
Predictions of thermalization of energy spectrum at large k (Roche 2013) at scales $k\ell \approx 1$ and larger.

- **Pressure spectrum**

Prediction of quantum nature of pressure spectrum (k^{-2} rather than the classical $k^{-7/3}$) due to singular nature of vorticity (Kivotides, CFB *et al* 2001).

- **Intermittency**

Predictions of deviations from $k^{-5/3}$ in temperature regime $\rho_n \approx \rho_s$ using shell model (Boué *et al* 2013)



Conclusions

- **Energy spectrum:**
  Mechanical drive: most energy is at small k with $E(k) \sim k^{-5/3}$, thermal drive $E(k)$ is a bump (lacks energy at small k) with $\sim k^{-1}$ tail.

- **Vortex line density spectrum**
  Probably $f^{-5/3}$ but less understood.

- **Much to do**
  about $k\ell \approx 1$ regime, low temperature regime, intermittency, pressure spectrum

References:

CFB, L'vov & Roche, PNAS 2014 (**review**)

Baggaley, Sherwin, CFB & Sergeev, PRB 2012

Sherwin, CFB & Baggaley, PRB 2015