# Chaotic blow-up scenarios in models and DNS

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# Shell models

# (Sabra) shell model of turbulence

Discrete sequence (geometric progression) of wavenumbers:

$$\lambda = 2: \quad k_0 = 1, \quad k_1 = 2, \quad k_2 = 4, \quad k_3 = 8, \quad k_4 = 16, \ldots$$

One complex variable $u_n$ describes velocity at the respective scale:

$$\left(\frac{d}{dt} + \nu k_n^2\right) u_n = i\left(k_{n+1} u_{n+2} u_{n+1}^* - \tfrac{1}{2} k_n u_{n+1} u_{n-1}^* + \tfrac{1}{2} k_{n-1} u_{n-1} u_{n-2}\right) + f_n$$

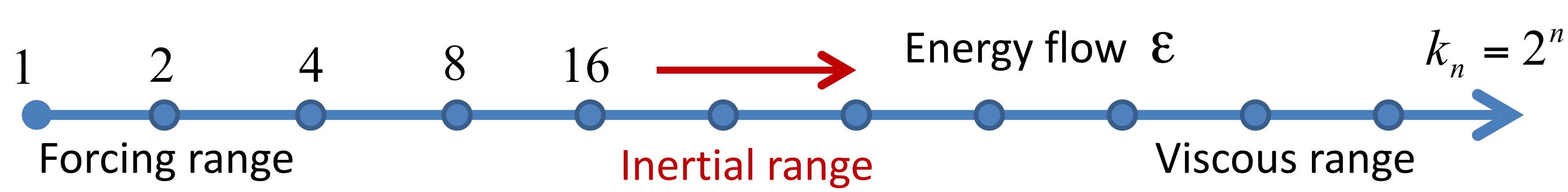
viscosity — quadratic convection term — forcing

(3D) inviscid invariants:

$$E = \frac{1}{2} \sum_n |u_n|^2 \qquad H = \sum_n (-)^n k_n |u_n|^2$$

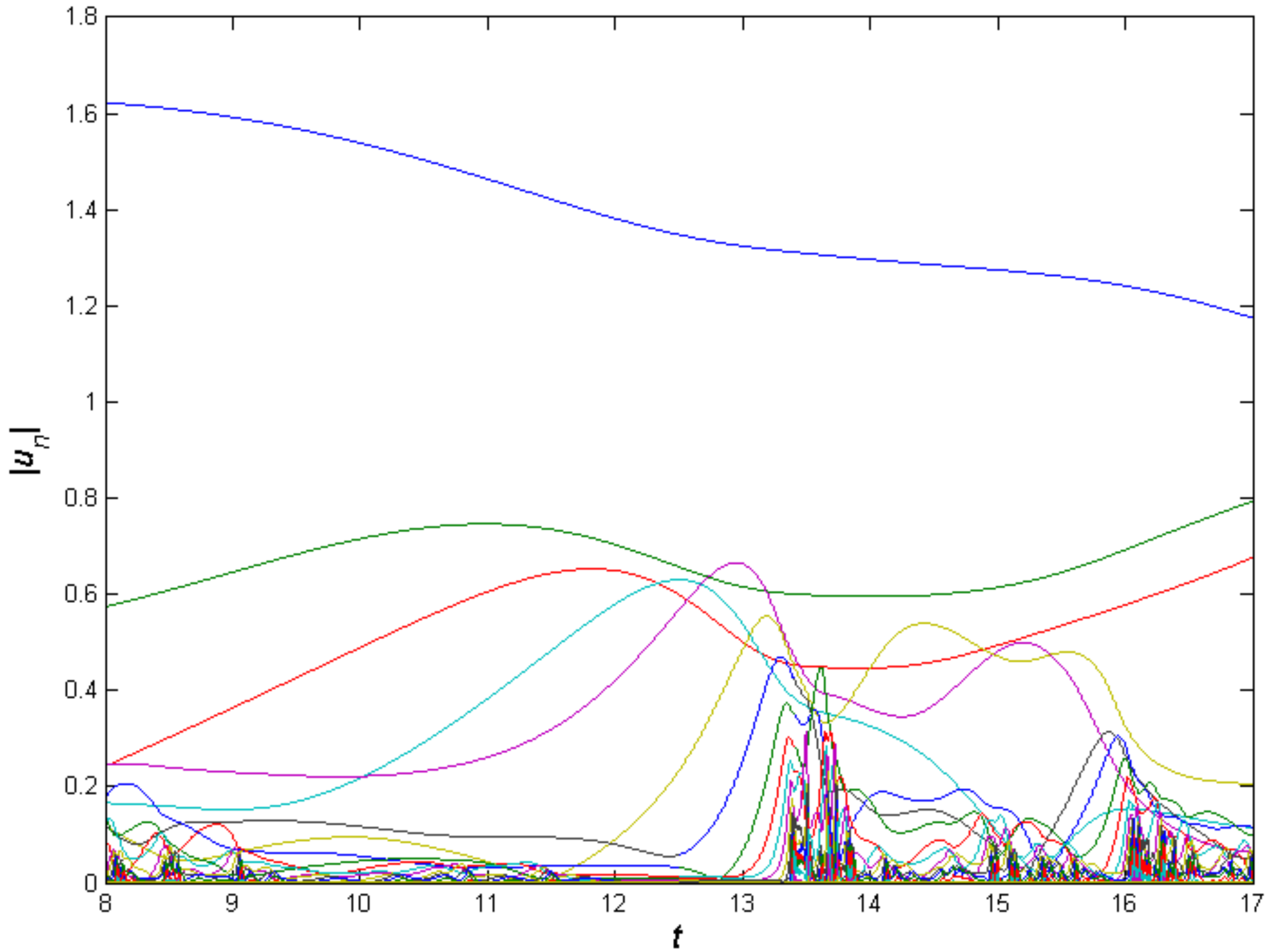
energy — helicity

Turbulent dynamics:



# Turbulence: shell model *vs.* 3D Navier-Stokes

typical temporal behavior




structure functions

$$S_p(k_n) = \left\langle |u_n|^p \right\rangle \propto k_n^{-\varsigma_p}$$

anomalous scaling deviating from K41 prediction $\zeta_p = p/3$

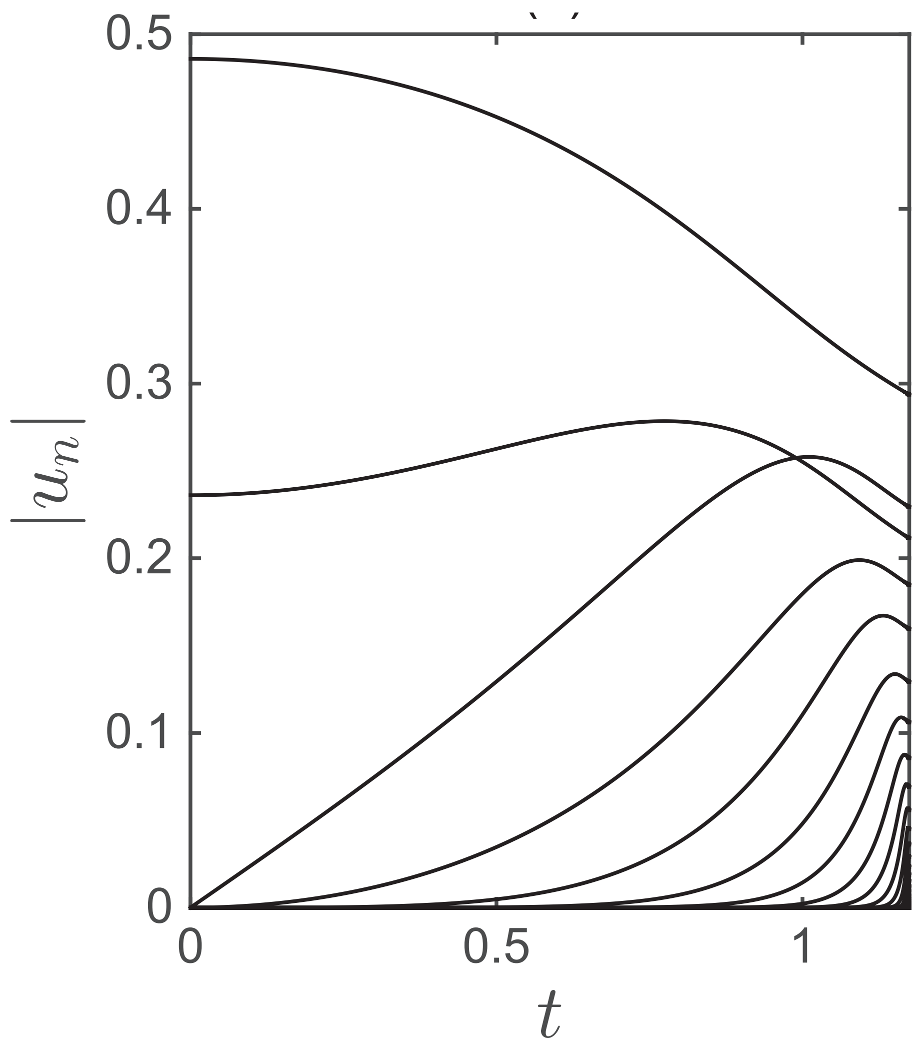
$$Sabra: \ \varsigma_2 = 0.72, \ \varsigma_3 = 1, \ \varsigma_4 = 1.26, \ \varsigma_5 = 1.49$$

$$NS: \ \varsigma_2 = 0.7, \ \varsigma_3 = 1, \ \varsigma_4 = 1.27, \ \varsigma_5 = 1.53$$

Model reproduces well the basic properties of the Navier-Stokes turbulence:

K41 theory, dissipative anomaly, intermittency, anomalous scaling

# Finite-time blowup: inviscid shell model *vs.* 3D Euler




self-similar asymptotic solution:

$$u_n(t) = -iu_* k_n^{-y_0} f[u_*(t_* - t)k_n^{1-y_0}]$$

(Dombre & Gilson, 1998)

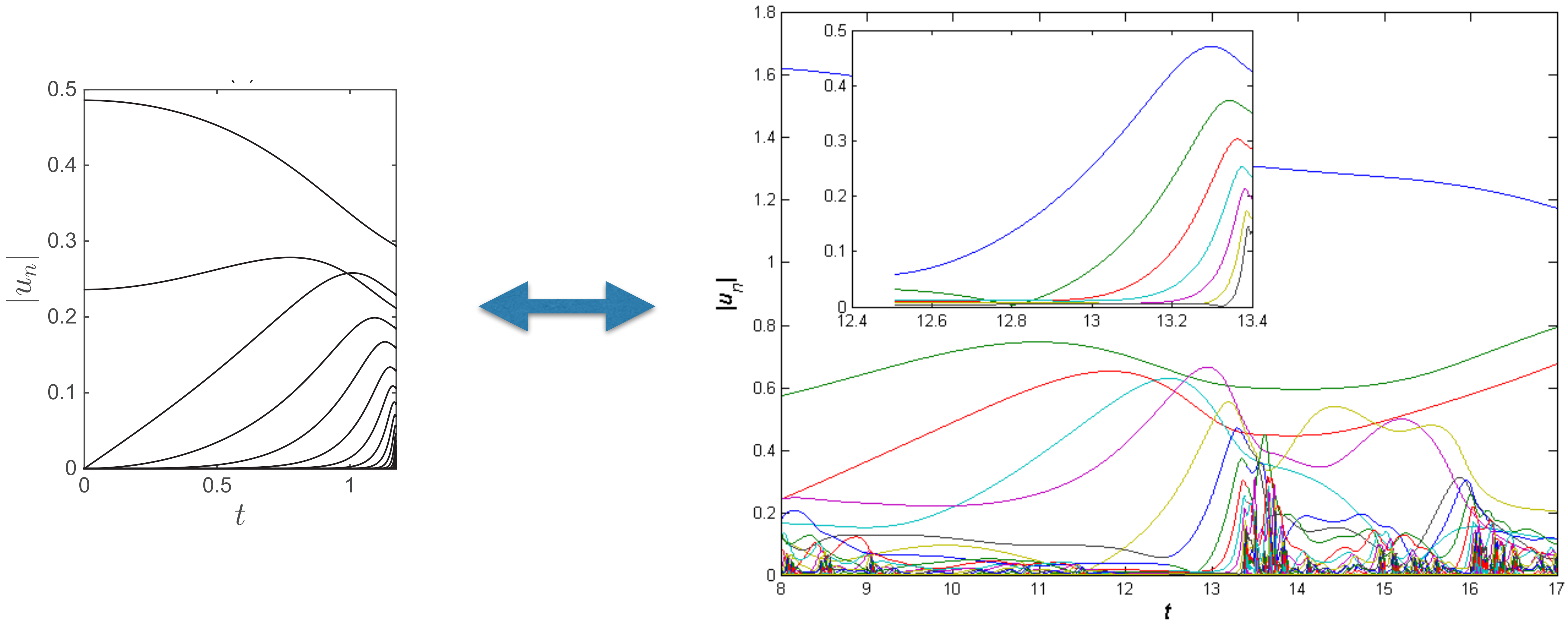
$$u_n \propto k_n^{-y_0}, \quad t_* - t \propto k_n^{1-y_0}$$

$$y_0 = 0.281$$

Self-similar blowup in
inviscid Burgers equation: $y_0 = 1/3$

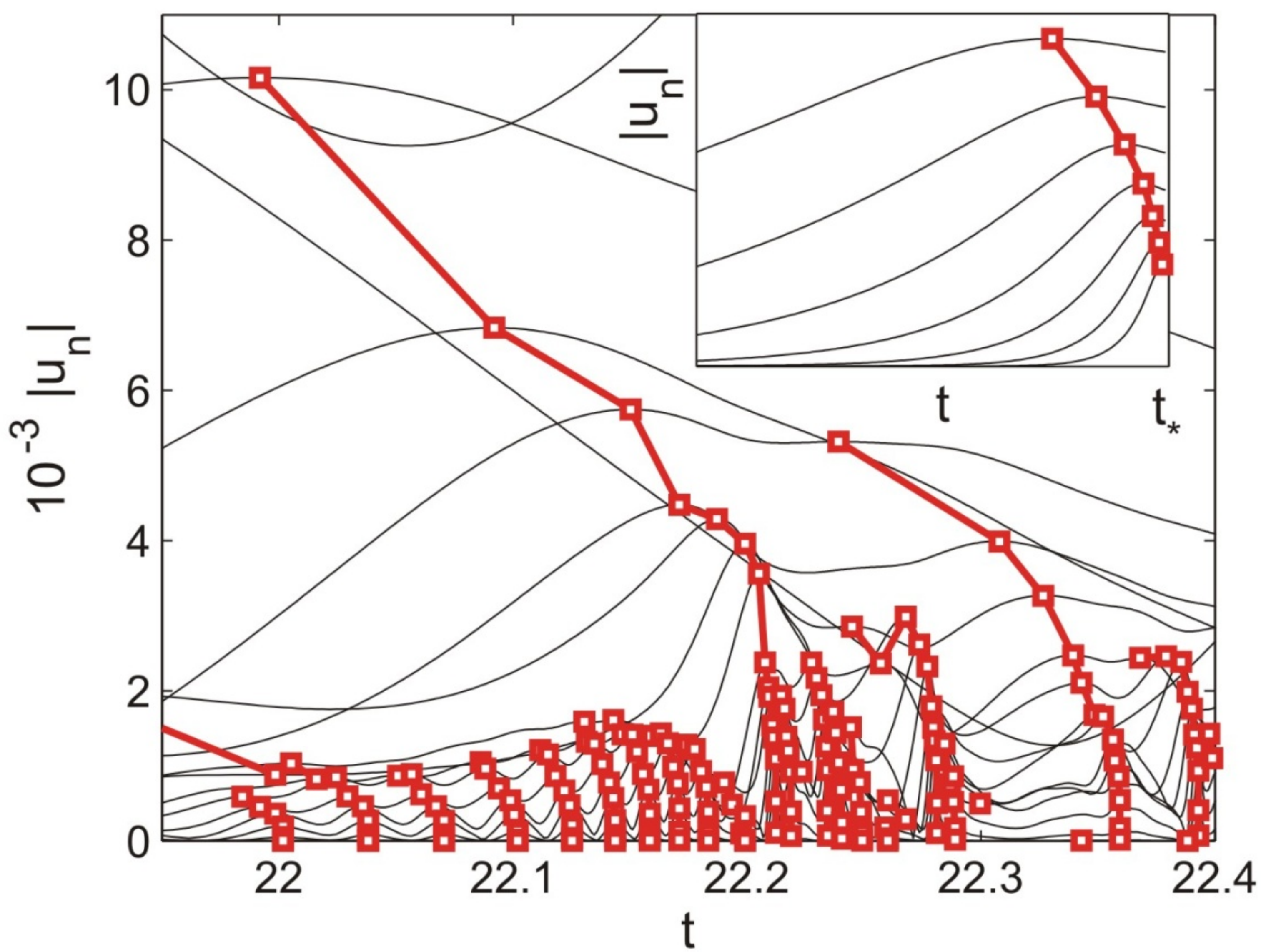
Model reproduces self-similar blowup of inviscid Burgers equation (compressible flow)

# Relation between blowup and turbulence?



# Instantons in turbulence for a viscous shell model

Simulations for 34 shells, Re ~ $10^{11}$




Instantons are identified as coherent structures using local maximums of shell velocities

$$v_n = \max_t u_n(t)$$

We consider only the **developed instantons**: they start at some shell $n_0$ and extend to the viscous range.

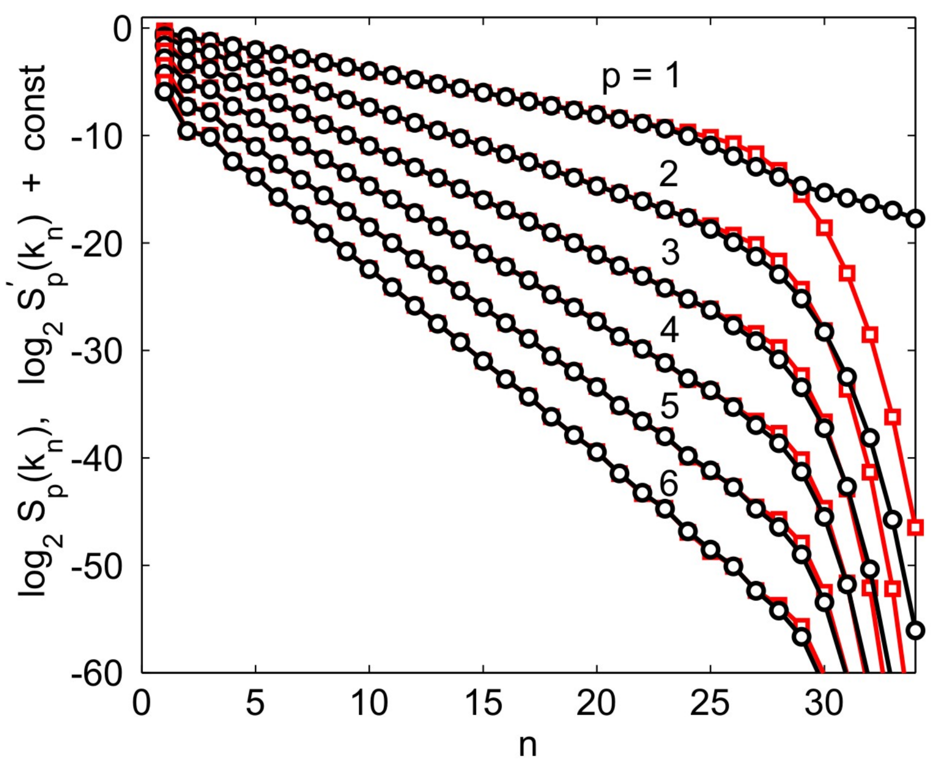
Developed instantons include 60-90% of all maxima at a given shell.

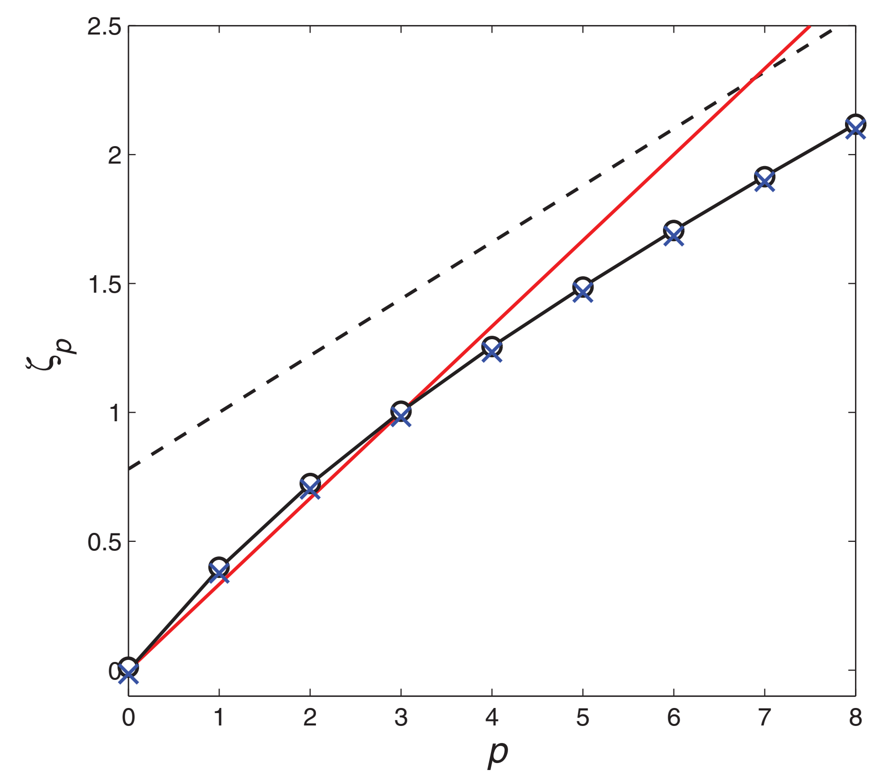
# Structure functions in terms of instantons

original definition: $S_p(k_n) = \left\langle |u_n|^p \right\rangle \propto k_n^{-\varsigma_p}$

new definition: $S'_p(k_n) = \lim_{T\to\infty} \frac{1}{Tk_n} \sum_{\text{all instantons}} v_n^{p-1}$

using the instanton lifetime as $t_n \approx (k_n v_n)^{-1}$



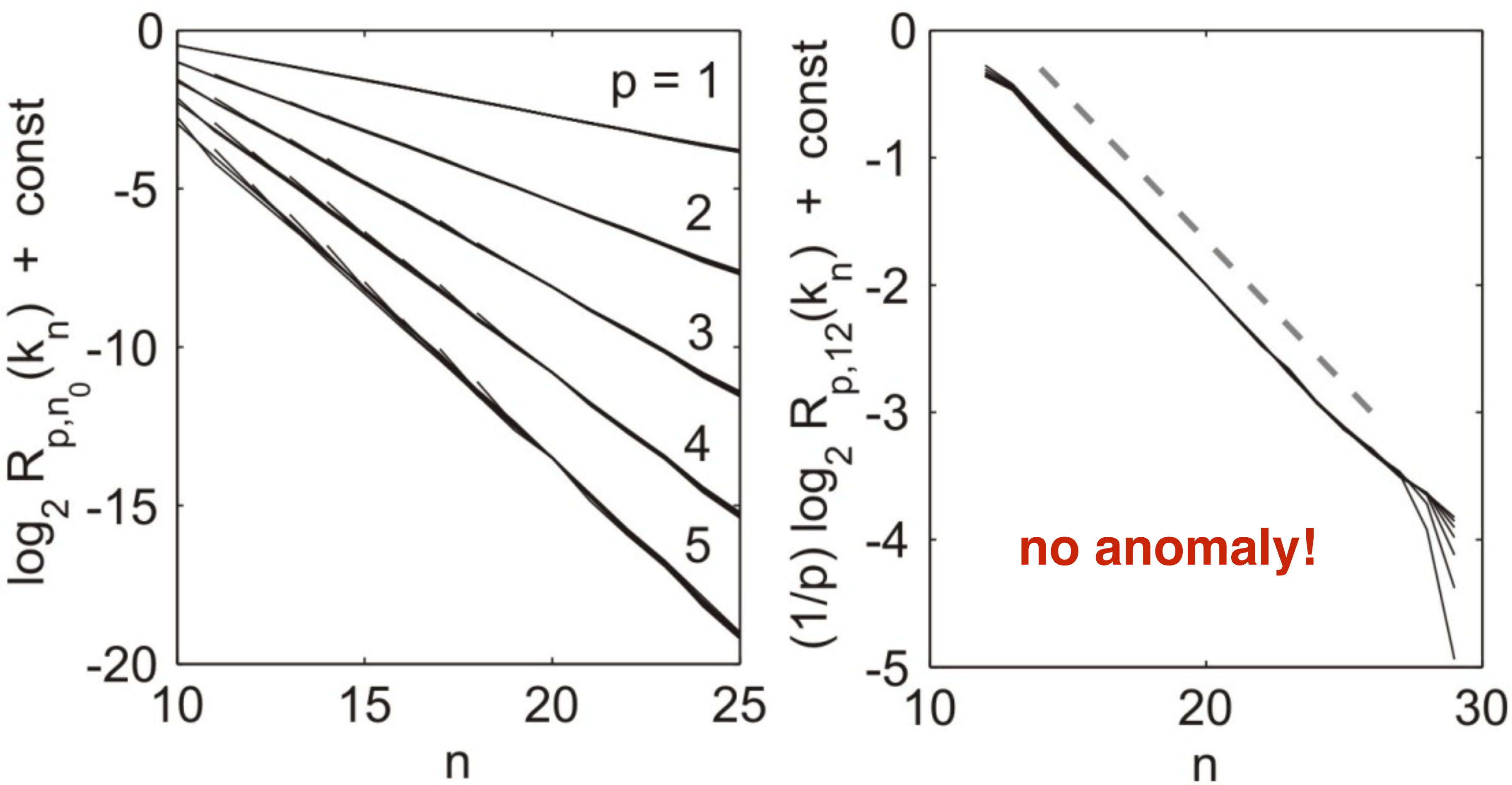

Same values of anomalous scaling exponents!

Intermittency of the instantons reproduces intermittency of the full system.

# Self-similar statistics of instants

**Proper** instanton structure functions: $R_{p,n_0}(k_n) = \lim_{T \to \infty} \frac{1}{T} \sum_{\substack{\text{all instantons} \\ \text{created in shell } n_0}} v_n^p \propto k_n^{-yp}$

$y \approx 0.22$




Instantons are statistically self-similar events with a **single universal** exponent

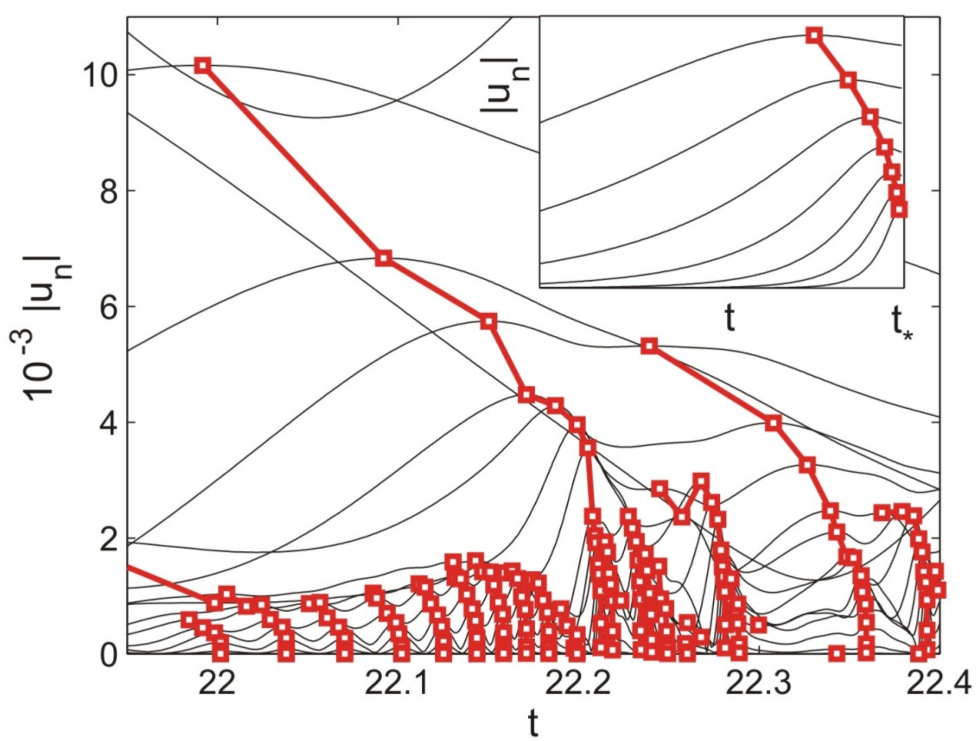
They can be seen as “dressed" blowup events.
The scaling exponent is modified due to interactions:
$y$ = 0.22 for instantons vs. 0.28 for blowup

# Instants within the multi-fractal picture of Parisi-Frisch

Buergers equation → Self-similar event: shock → Intermittency: bifractal picture defined by scaling of **single** shock

shell model → Statistically self-similar instantons → Intermittency: multi-fractal picture defined by **distribution** of instantons

intermittency does not require events with different scaling exponents, contrary to the suggestion of Parisi (1990)



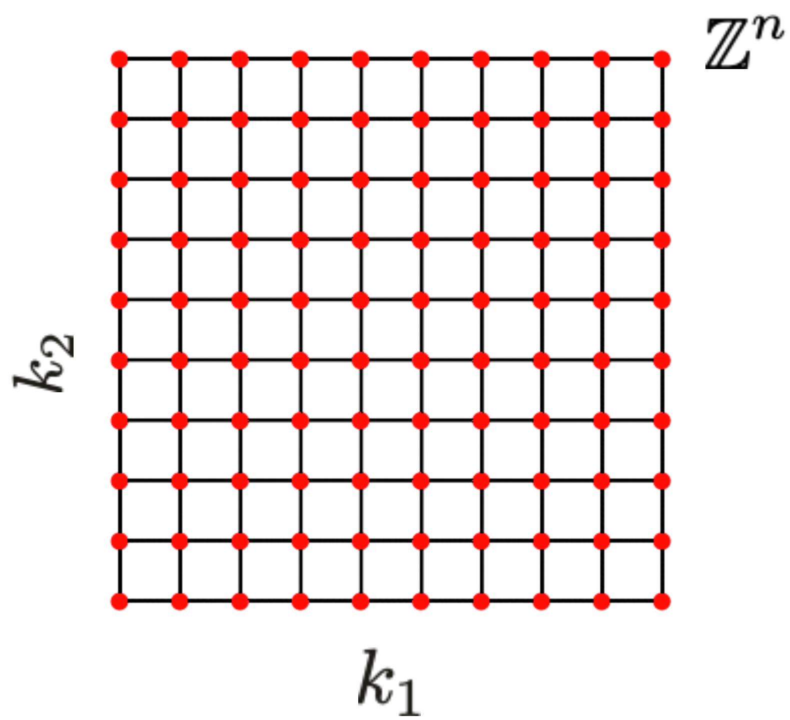

Navier-Stokes equations → ?

Models on logarithmic lattices
(with Ciro Campolina)

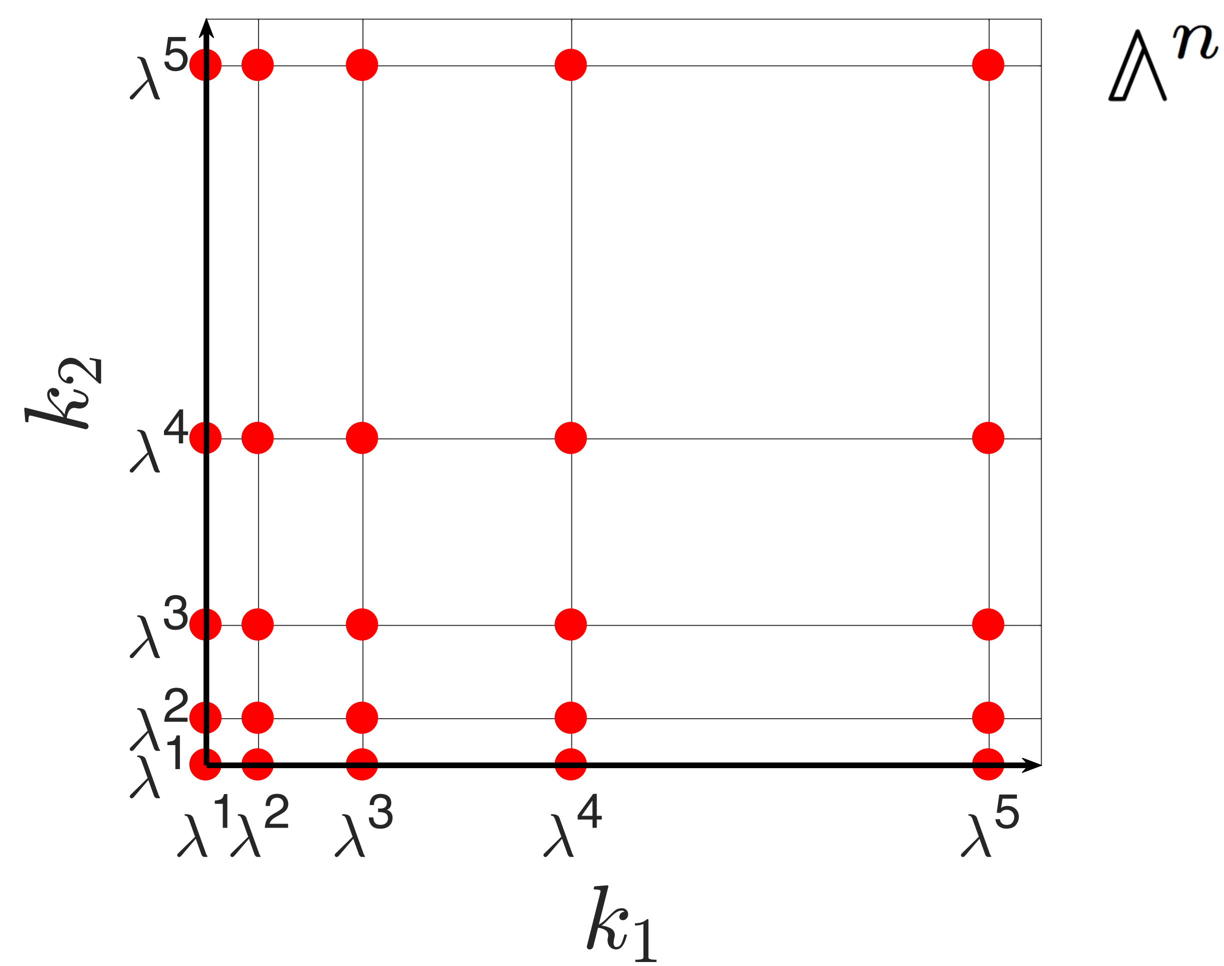
# Proposed technique

Modify space, not equations!

**Periodic domain**




**Log model**



# Algebraic structure on a logarithmic lattice

Function: $f(\mathbf{k}) \in \mathbb{C}, \quad \mathbf{k} \in \Lambda^n$ (Fourier space)

Sum: $(f+g)(\mathbf{k}) = f(\mathbf{k}) + g(\mathbf{k})$ Scalar product: $(f,g) := \sum_{\mathbf{k}\in\Lambda^n} f(\mathbf{k})\overline{g(\mathbf{k})}$

Derivative: $\partial_j f(\mathbf{k}) = ik_j f(\mathbf{k})$

Product: $(f * g)(k)$

*(P.1)* *(Reality condition)* $(f*g)(-\mathbf{k}) = \overline{(f*g)(\mathbf{k})}$;

*(P.2)* *(Bilinearity)* $(f+\gamma g)*h = f*h+\gamma(g*h)$, for any number $\gamma \in \mathbb{R}$;

*(P.3)* *(Commutativity)* $f*g = g*f$;

*(P.4)* *(Associativity in average)* $(f*g,h) = (f,g*h)$;

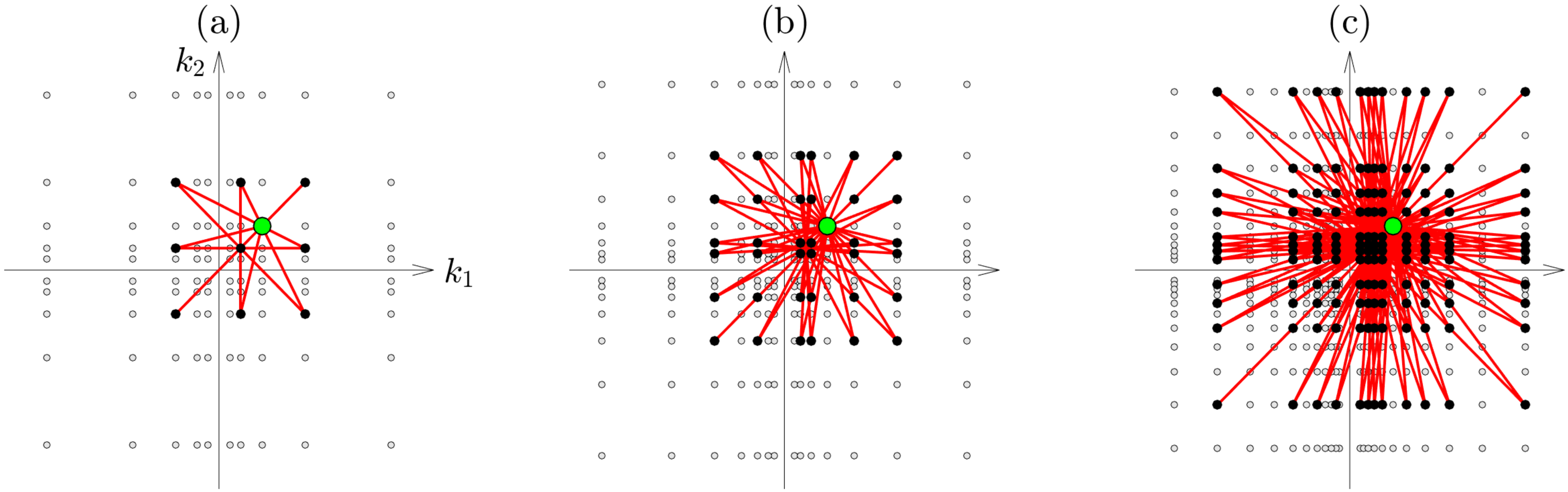
*(P.5)* *(Leibniz rule)* $\partial_j(f*g) = \partial_j f * g + f * \partial_j g$, for $j = 1,\ldots,d$;

**triads**
$\mathbf{k} = \mathbf{p} + \mathbf{q}$

additionally one can ask for symmetries:
translation and scaling invariance, isotropy

# Three types of lattices

$$\mathbf{k} = \mathbf{p} + \mathbf{q}$$




$$\lambda = 2$$

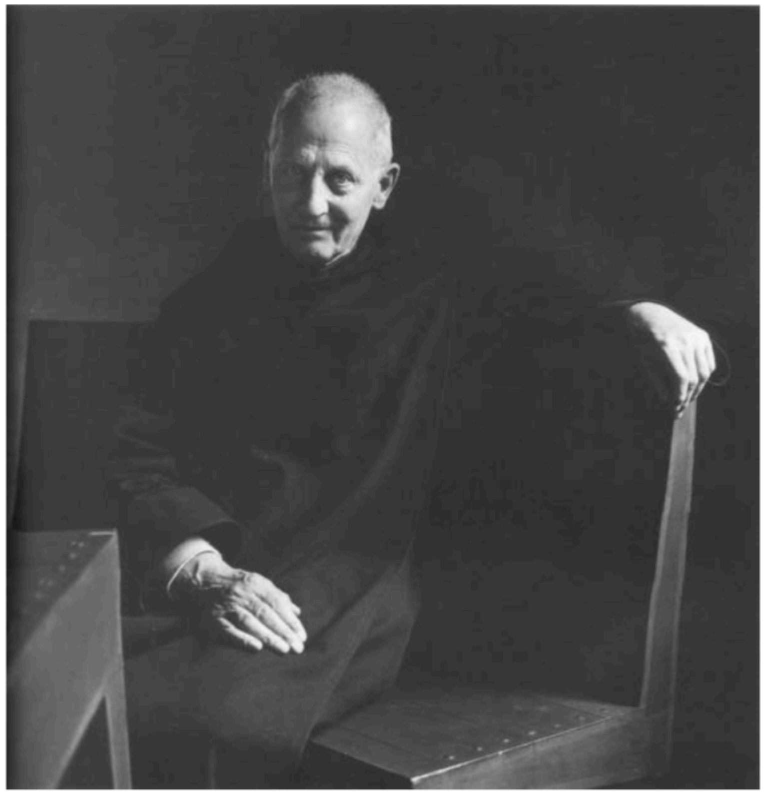
$$\lambda = \frac{1+\sqrt{5}}{2} \approx 1.62$$

**golden mean**

$$\lambda = \frac{\sqrt[3]{9+\sqrt{69}} + \sqrt[3]{9-\sqrt{69}}}{\sqrt[3]{18}} \approx 1.325$$

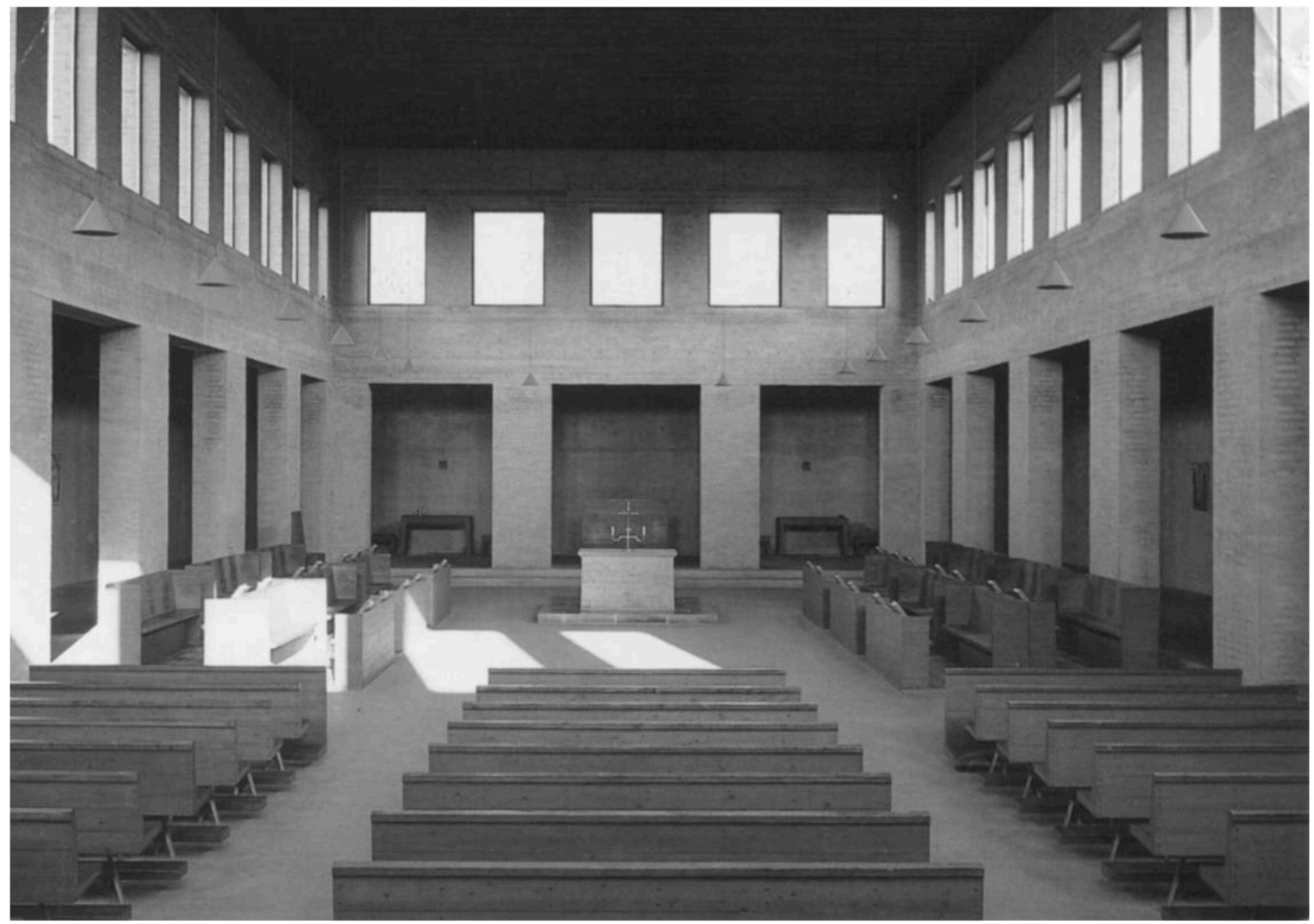
**plastic number**

# Plastic number



*The Benedict monk and architect*
*Dom Hans van der Laan (1904—1991)*

$$\sigma = \frac{\sqrt[3]{9+\sqrt{69}} + \sqrt[3]{9-\sqrt{69}}}{\sqrt[3]{18}} \approx 1.325$$



*St. Benedictusberg Abbey, at Mamelis*
*(Vaals, Netherlands)*

# Modeling on logarithmic lattices

Product: $(u * v)(\mathbf{k}) = \sum_{\substack{\mathbf{p},\mathbf{q} \in \Lambda^n \\ \mathbf{k}=\mathbf{p}+\mathbf{q}}} u(\mathbf{p}) v(\mathbf{q})$

all possible product can be classified

Associativity issue:

$$(f * g) * h \neq f * (g * h)$$
$$(f * g, h) = (f, g * h)$$

**associative in average**

We can construct an **analogue of any PDE** on a logarithmic lattice that has **quadratic nonlinearities**: **take equations written in the same way**.

It will automatically conserve **linear, quadratic and cubic conservation laws**.

Examples: Navier-Stokes and Euler equations, MHD, Boltmann equations etc.

$$\partial_t u_i + u_j * \partial_j u_i = -\partial_i p, \quad \partial_j u_j = 0,$$

(incompressible Euler equations)

**Invariants**:

- Energy: $E = \frac{1}{2}\langle u_j, u_j \rangle$
- Helicity: $H = \langle u_j, \omega_j \rangle$
- Kelvin's theorem: **infinite number** of invariants (cross correlation conservation)

# 1D Burgers equation

$$\partial_t u + u * \partial_x u = \nu \partial_x^2 u$$

$\lambda = 2$

**Desnyansky-Novikov (dyadic) model**

**Self-similar blowup**

K41 solution (**shock wave**)

$\lambda = (1+\sqrt{5})/2$

**Sabra model**

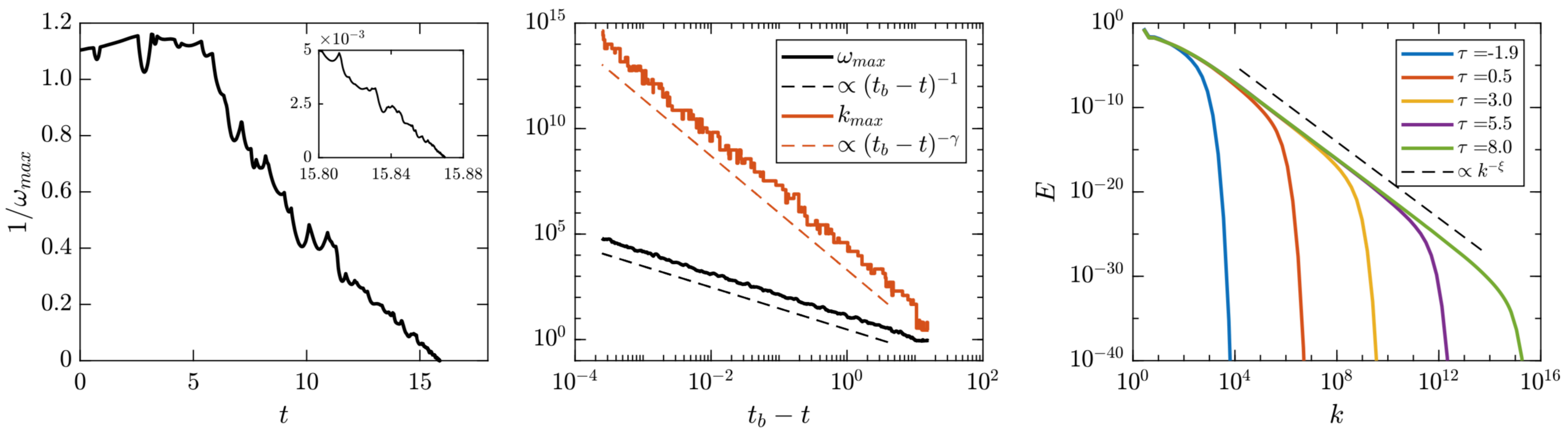
(for some values of parameters)

# 3D incompressible Euler equations

**Open problem: can singularity form in finite time?**

$$\partial_t u_i + u_j * \partial_j u_i = -\partial_i p, \quad \partial_j u_j = 0,$$

- **Adaptive time stepping** RKF4(5): relative local error for $\omega_{\max}$ was kept below $10^{-10}$
- **Adaptive number of nodes N**: error for enstrophy $\Omega = \frac{1}{2}\langle \omega_j, \omega_j \rangle$ below $10^{-20}$
- 13180 total time steps




$$\lambda = (1 + \sqrt{5})/2$$

$$\omega_{\max} \sim (t_b - t)^{-1}$$

$$k_{\max} \sim (t_b - t)^{-\gamma}$$

$$\gamma = 2.70$$

**Spatial range of** $k \approx 10^{16}$

$$E \propto k^{-\xi}$$

$$\xi = 3 - 2/\gamma = 2.26$$

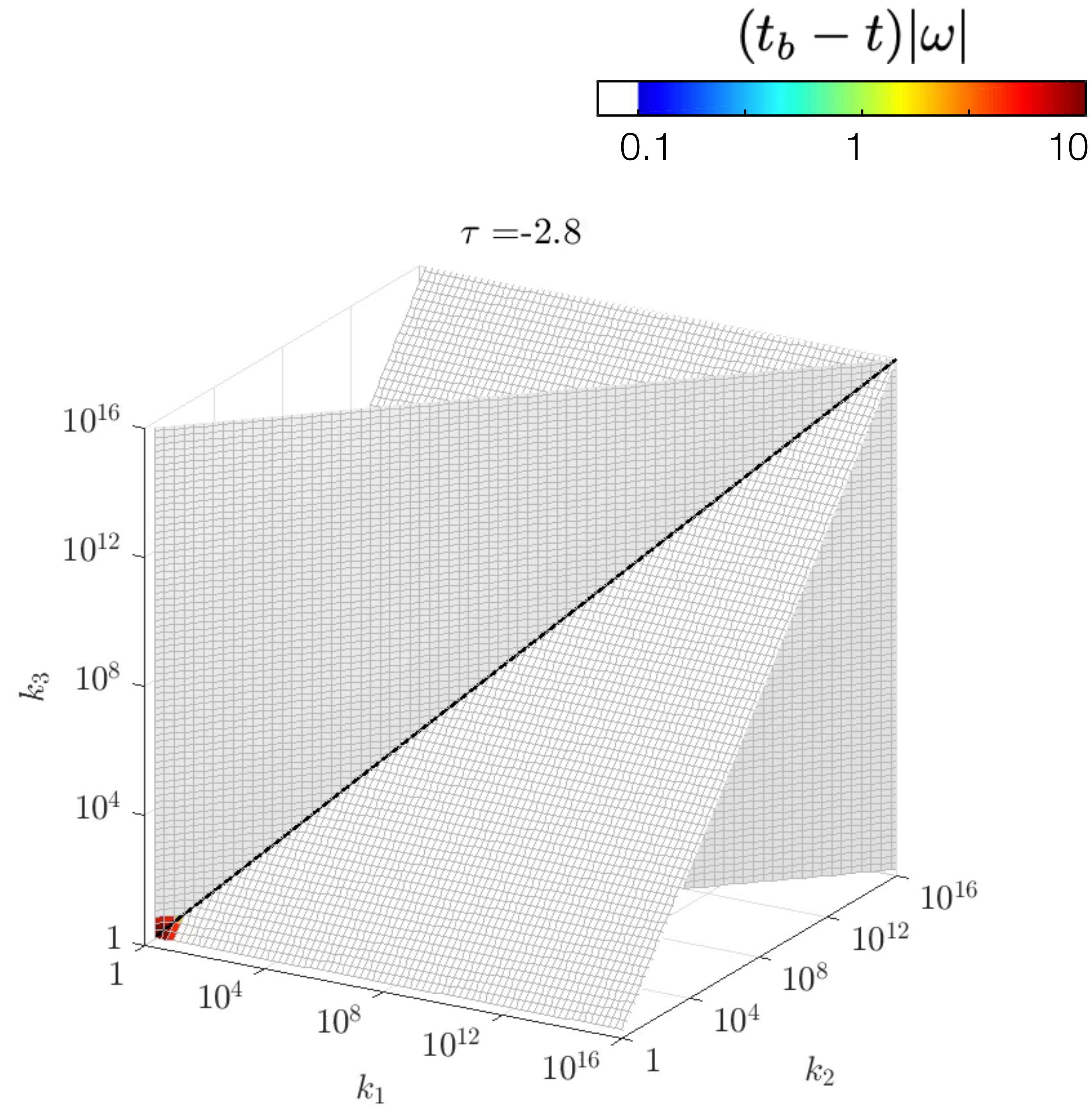
# 3D incompressible Euler equations

Traveling **chaotic wave** attractor
in the properly renormalised system

$$\tau = -\log(t_b - t)$$

Maximum Lyapunov exponent:

$$\lambda_{\max} = 9.18$$




CC & AM, PRL **121**, 064501 (2018).

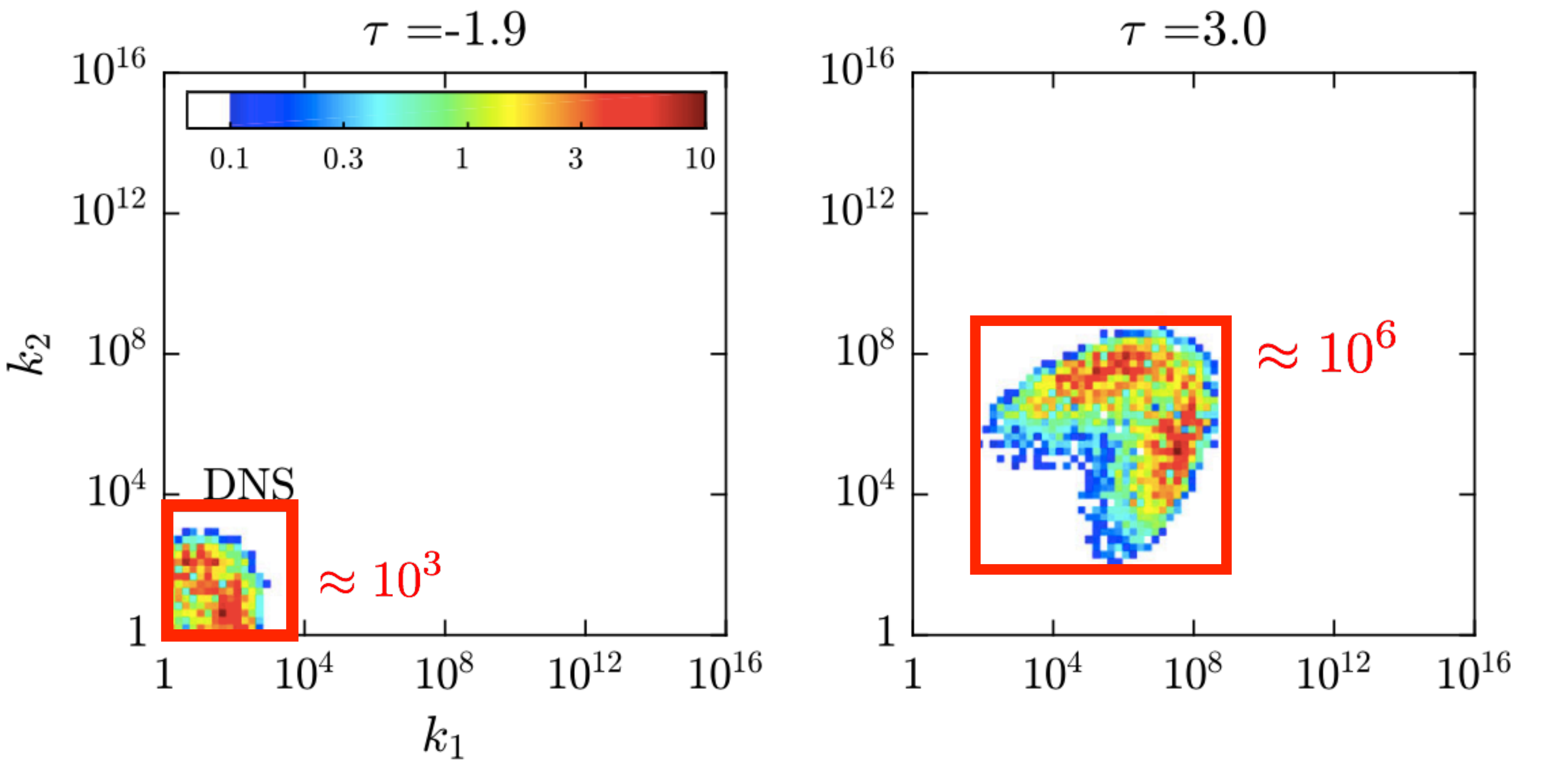
Vasseur&Vishik 2019: Euler blowup
must be sensitive to initial conditions

## Can the blowup be observed with DNS?

**State-of-the-art DNS:**

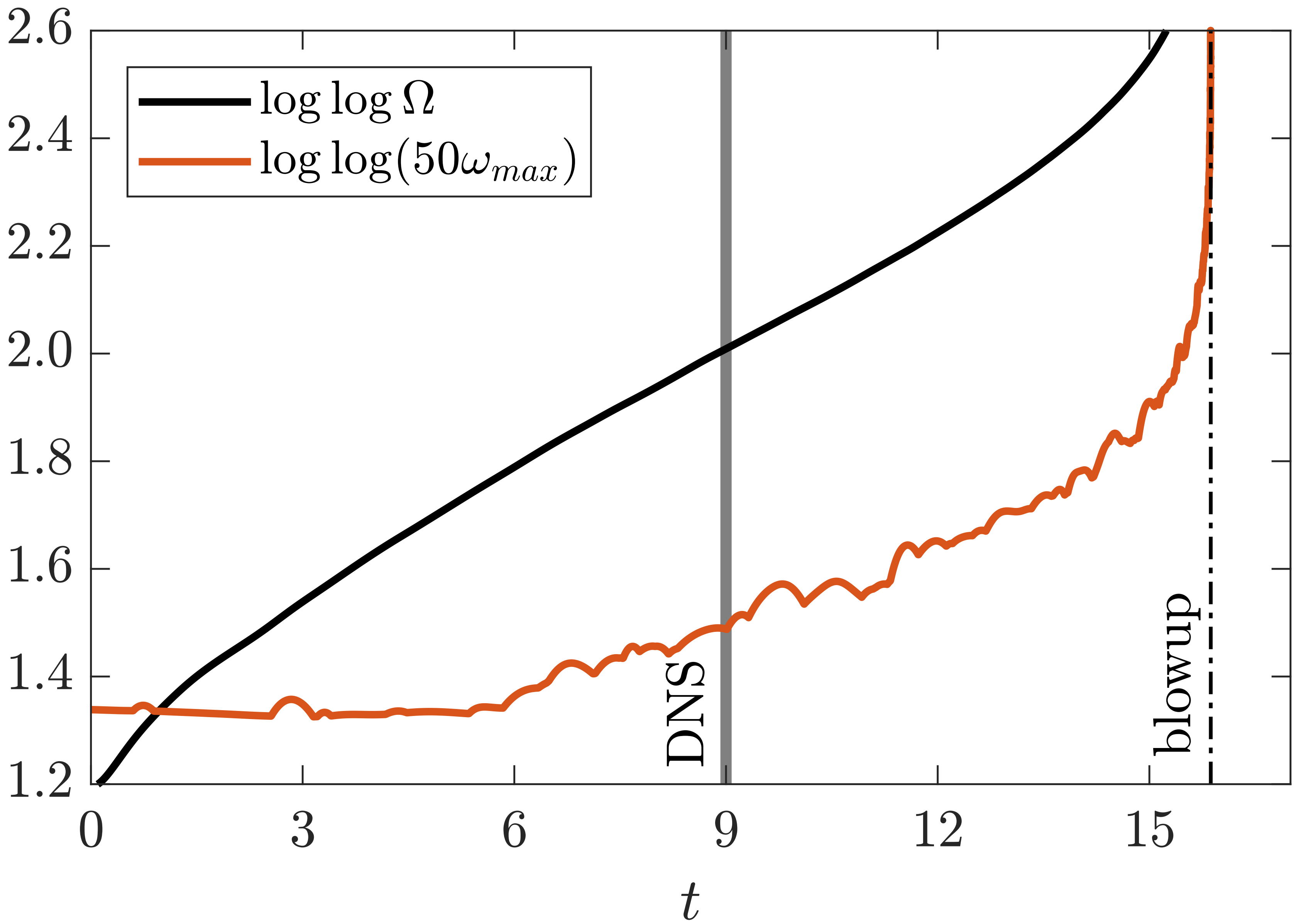
Max wave number $\leq 4 \times 10^3$

**Size of attractor:** $\approx 10^6$




**Such blowup cannot be observed with state-of-the-art DNS!**

**How about experiments?**

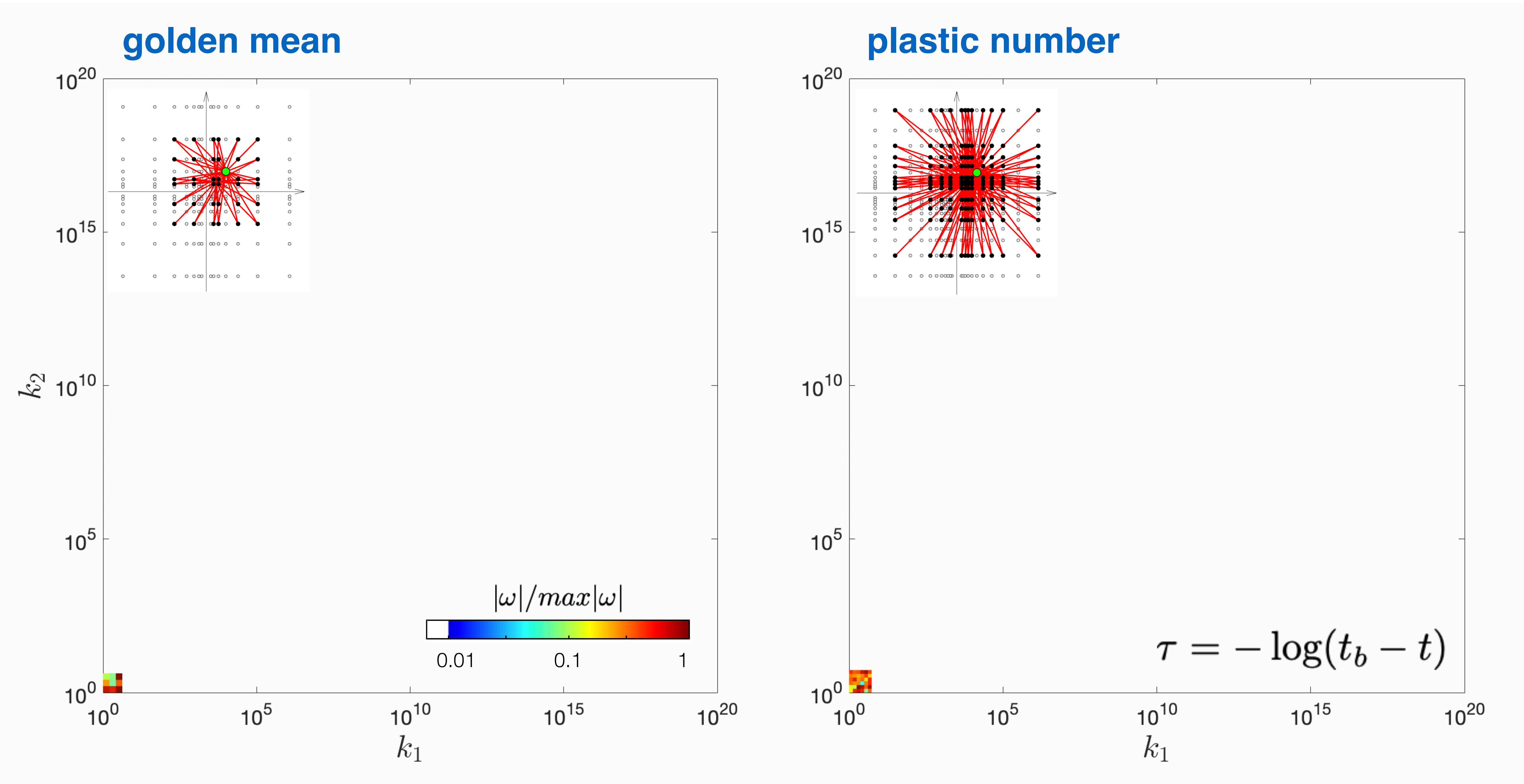



At DNS scales, the growth is not greater than double exponential — in accordance with KERR ‘13; HOU ‘09

# How robust is the chaotic blowup?

## Blowup in ideal 2D Boussinesq equations (buoyancy)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = -\frac{1}{\rho}\nabla p \;-\mathbf{g}\alpha\Delta T \qquad \frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \mathbf{0}$$




**Difference in scaling laws is about 5%**

# Conclusions

- Relation between blowup and intermittency for shell models of turbulence

intermittency is as in the Navier-Stokes turbulence

blowup as in the inviscid Burgers equation

intermittency is controlled by a distribution
of statistically **self-similar** events (instantons),
having a **single** scaling exponent

- Blowup for 3D incompressible Euler equations on a logarithmic lattice

use exactly the same equation of motion

automatically preserved structure, symmetries, invariants

blowup is a **chaotic** wave with a core of about 6 decades

modern DNS capabilities may be **by far** insufficient to see it

References: PRE 86, 025301(R); PRE 87, 053011; PRL 121, 064501

Thank you!

alexei.impa.br