The Painlevé property, one century later

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"Les fonctions, comme les êtres vivants, sont caractérisées par leurs singularités", used to say the mathematician Paul Montel. In order to define a function, a differential equation must have a general solution without any branching around the singular points whose position depends on the initial conditions, this is called the Painlevé property. Linear equations all have this property and provide the everyday elements for building analytic expressions: polynomials, exponentials, trigonometric functions, Bessel, Legendre, ... Nonlinear equations, on the contrary, provide very few (new) functions. For instance, first order only defines one new function, the elliptic function, i. e. the famous inverse function $u(x)$ of the so-called elliptic integral $x(u)$ ruling the motion of the pendulum.

This is the systematic examination of all differential equations of second order which in the period 1898–1906 led Painlevé and his student Gambier to the discovery of six new functions defined by a differential equation, see Fig. 1, the first one (P1) being as simple as $\mathrm{d}^2u/\mathrm{d}x^2 = 6u^2+x$. Initially qualified by Poincaré as "île originale et splendide dans l'océan voisin [du continent des mathématiques]", i.e. without any relation with the rest of science, this feat has for long only excited the curiosity of a few mathematicians (mainly from Belgium, Japan and Russia), since the fashion turned to be more abstract mathematics.

The renewal of interest dates backs to the end of the sixties, pushed by three apparently distinct domains of physics, thus taking the relay of mathematics. Firstly, the discovery of the soliton by Zabusky and Kruskal in 1965 [2] has initiated the booming domain of integrable evolution equations, i. e. those admitting elastic collisions of quasi-particles called multi-solitons, see Fig. 2, (for a review, see [9]). The link with Painlevé [3] is that, whenever such an integrable equation admits what is technically called a noncharacteristic reduction to an ordinary differential equation (ODE), this ODE has the Painlevé property; one such example is the Korteweg-de Vries equation for $U(X,T)$

$$U_T + U_{XXX} - 12UU_X = 0,$$

and its reduction to (P1) for $u(x)$

$$u(x) = U(X,T) + T,\ x = X - 6T^2.$$

Secondly, statistical physics and field theory have provided a number of exactly solvable models, in which typically the free energy or some correlation functions, as functions of say the temperature, would obey one of the six Painlevé equations. A celebrated example is the two-dimensional Ising model [4]. The self-dual Yang-Mills equations, fundamental in field theory, admit a reduction to the master Painlevé equation (P6) [10].

Thirdly, the advent of computers has popularized the observation that dynamical systems such as the Lorenz model of atmospheric circulation,

$$x' = \sigma(y - x),\ y' = -xz + rx - y,\ z' = xy - bz,$$

although deterministic i.e. first order in time, generically exhibit a chaotic behaviour. However, for some values of the control parameters (b, σ, r), easily found by what is called the Painlevé test, they may have the Painlevé property; for instance, the case $b = 0, \sigma = 1/3$ is integrable with (P3) [5].

This renewal has produced a feedback in two major directions: the extension of the theory of Painlevé to partial differential equations [6] and more recently to finite difference equations under the impulse of lattice statistical physics [8].

In our youth, we all learnt quite many recipes to integrate nonlinear differential equations. This is not the way, and the simple and powerful methods to achieve the goal when this is possible have been developed one century ago in the famous *Leçons de Stockholm* [1] delivered by Painlevé at the invitation of His Majesty the king of Sweden and Norway. This text, although (temporarily?) out of print, is still a reference and the anniversary will be marked by a three-week school at Cargèse this spring, teaching both the old but not obsolete at all methods and the modern extensions to physics. It will be honoured by the presence of Professeur Bureau (Liège), the last true disciple of Painlevé school, and Professor Kruskal (Rutgers), the co-discoverer of the soliton.

To close with an industrial touch, the signal which propagates in an optical fiber on the floor of the oceans is a solitary wave, with an exact analytic expression depending on very few arbitrary, i.e. adjustable parameters, and tremendous efforts are necessary to amplify the signal every about 400 km. The present theory predicts the probable existence of an exact solution with more free parameters; its discovery, a difficult task, will necessarily involve the present methods and it will bring a considerable progress to the question of attenuation.

Figure 1. Solutions $u(x)$ of the second Painlevé equation (P2) $\mathrm{d}^2u/\mathrm{d}x^2 = 2u^3 + xu + \alpha$, in the case $\alpha = 0$, vanishing for $x \to +\infty$. These are asymptotic to a multiple $k\,\mathrm{Ai}(x)$ of the Airy function $\mathrm{Ai}(x)$. (Courtesy of Cambridge University Press ref. [9]).

Figure 2. Elastic collision of two solitons in the Korteweg-de Vries equation. (Courtesy of Cambridge University Press ref. [9]).

Tutorial introductions. See ref. [7, 11].

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