BEYOND THE TWO–SINGULAR MANIFOLD METHOD[a]

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For its coherence, the Painlevé approach must find the Bäcklund transformation (BT) of a partial differential equation (PDE) when there is one. For the AKNS system, both the one–singular manifold method of Weiss and the two–singular manifold method fail to retrieve the BT. We find it by combining the point symmetries of the PDE and the one–singular manifold method.

1 Introduction

When a PDE satisfies some necessary conditions ("passes the Painlevé test" [24,15]) for the absence of movable critical singularities in its general solution (the "Painlevé property" (PP)), the next step is to find a BT [2,20], in order to prove the sufficiency of these conditions. A BT between two given PDEs

$$E_1(u,x,t)=0,\ E_2(U,X,T)=0 \tag{1}$$

is by definition ([8] vol. III chap. XII, [13]) a pair of relations

$$F_j(u,x,t,U,X,T)=0,\ j=1,2 \tag{2}$$

in which F_j depends on the derivatives of $u(x,t)$ and $U(X,T)$, such that the elimination of u (resp. U) between (F_1,F_2) implies $E_2=0$ (resp. $E_1=0$). In case the PDEs are the same, the BT is called the auto-BT.

For instance, the AKNS system [26,1]

$$E^{(1)} \equiv iu_t + p_r u_{xx} + q_r u^2 v = 0,\ E^{(2)} \equiv -iv_t + p_r v_{xx} + q_r u v^2 = 0 \tag{3}$$

has the BT [10,3,9,12] ($a^2=-2p_r/q_r,\ R=\pm\sqrt{(u+U)(v+V)/a^2-(\lambda-\mu)^2}$)

$$\begin{aligned}(u+U)_x &= -(u-U)R - i(\lambda+\mu)(u+U)\\ (v+V)_x &= -(v-V)R + i(\lambda+\mu)(v+V)\end{aligned}$$

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$$\begin{aligned} +ip_r^{-1}(u+U)_t &= (u-U)_x R + (u+U)M + i(\lambda+\mu)(u+U)_x \\ -ip_r^{-1}(v+V)_t &= (v-V)_x R + (v+V)M - i(\lambda+\mu)(v+V)_x \\ M &= (uv+UV)/a^2 \end{aligned} \tag{4}$$

λ and μ being arbitrary complex constants. Galilean invariance $(x,t,u,v) \to (x-2p_r ct, t, e^{i(cx-p_r c^2 t)}u, e^{-i(cx-p_r c^2 t)}v)$ allows to choose $c=\lambda+\mu=0$ [9].

Up to now, the above BT has been retrieved neither by the one–singular manifold method [23], nor by the two–singular manifold method [16]. This is a challenge to the Painlevé approach, which for most PDEs succeeds to find the BT by singularity analysis *only*. We do so in Section 6.

2 The ideas. Darboux, Lax and Bäcklund

For the six ordinary differential equations which bear his name, Painlevé proved the PP by showing [17,18] the existence of one (case of (P1)) or two ((P2)–(P6)) function(s) ψ linked to the general solution u by logarithmic derivatives

$$(\mathrm{P1}): \quad u = \mathcal{D}_1 \operatorname{Log} \psi \tag{5}$$

$$(\mathrm{P}n),\ n=2,\dots,6: \quad u = \mathcal{D}_n(\operatorname{Log}\psi_1 - \operatorname{Log}\psi_2) \tag{6}$$

where the operators $\mathcal{D}_n$ are linear (α is a constant):

$$\begin{aligned} &\mathcal{D}_1 = -\partial_x^2,\ \mathcal{D}_2 = \mathcal{D}_4 = \pm\partial_x, &(7)\\ &\mathcal{D}_3 = \pm e^{-x}\partial_x,\ \mathcal{D}_5 = \pm x e^{-x}(2\alpha)^{-1/2}\partial_x,\ \mathcal{D}_6 = \pm x(x-1)e^{-x}(2\alpha)^{-1/2}\partial_x. \end{aligned}$$

These functions ψ_1, ψ_2 have the same kind of singularities than solutions of *linear* ODEs, namely: they are entire functions for (P1)–(P5), and their only singularities for (P6) are three fixed critical points.

For PDEs, the analog of (5)–(6) is the *Darboux transformation* [7]

$$\mathrm{DT}: \ u = \sum_f \mathcal{D}_f \operatorname{Log}\psi_f + U \tag{8}$$

linking two solutions (u,U) of the PDE *via* logarithmic derivatives of scalars ψ_f attached to families f of movable singularities. Operators $\mathcal{D}_f$ can be derived by the Painlevé test. The ψ_f's satisfy a system of linear PDEs, the *Lax pair* [11]

$$\text{Lax pair}: \ L_1(U,\lambda)\Psi = 0,\ L_2(U,\lambda)\Psi = 0,\ \Psi = \mathrm{col}(\psi_f) \tag{9}$$

with coefficients depending on the second solution U and possibly some arbitrary constant λ, and the property that the vanishing of the commutator $[L_1, L_2]$ is equivalent to the PDE $E(U)=0$. The *Bäcklund transformation*

$$\mathrm{BT}: \ F_1(u,U,\lambda) = 0,\ F_2(u,U,\lambda) = 0 \tag{10}$$

results from the elimination [22] of Ψ between the DT (8) and the Lax pair (9).

3 Weiss method and its limitations

Let $\chi = 0$ be the equation for a given family of movable singularities, $-p$ and $-q$ the two positive integers equal to the singularity orders of u and $E(u)$. The assumption of the existence of a finite expression [24]

$$u_T = \sum_{j=0}^{-p} u_j \chi^{j+p} \tag{11}$$

implies

$$E_T = E(u_T) = \sum_{j=0}^{-q} E_j \chi^{j+q}, \tag{12}$$

where coefficients (u_j, E_j) only depend on the homographic invariants (S, C) occuring in the gradient of χ [4]

$$\chi_x = 1 + \frac{S}{2}\chi^2,\ \chi_t = -C + C_x \chi - \frac{1}{2}(CS + C_{xx})\chi^2 \tag{13}$$

and the arbitrary coefficients u_i of the Fuchs indices i in the interval $[0, -p]$. This “truncation $(p:0)$” (11) only exists if there is a nonempty solution to the set of overdetermined equations, called *Painlevé–Bäcklund equations* (PB),

$$\forall j \in]-p, -q] :\ E_j(S, C, \{u_i, i \text{ Fuchs index } \in [0, -p]\}) = 0. \tag{14}$$

In the case of Korteweg-de Vries (KdV) equation $(p = -2, q = -5)$

$$(\text{KdV}):\ E \equiv u_t + \left(u_{xx} - (3/a)u^2\right)_x = 0,\ a \text{ constant}, \tag{15}$$

the method is successful (see details ref. [16] section 2): the second solution U and the scalar solution ψ of a linear system are

$$U = -a(C + 2S)/6,\ \operatorname{Log}\psi = \int \chi^{-1} \mathrm{d}x, \tag{16}$$

resulting in the Darboux transformation

$$u_T = -2a\partial_x^2 \operatorname{Log}\psi + U; \tag{17}$$

the spectral parameter λ is $(C-S)/6$, and the Lax pair is the linearized version of the Riccati system for χ

$$(\chi^{-1})_x = -\chi^{-2} + U/a + \lambda,\ (\chi^{-1})_t = ((2U/a - 4\lambda)\chi^{-1} - U_x/a)_x. \tag{18}$$

The BT then results from the elimination of χ between (17) and (18):

$$\chi^{-1} = -(w-W)/(2a),\ u_T = u = w_x,\ U = W_x \quad (19)$$

$$a(w+W)_x = -2a^2\lambda + (w-W)^2/2 \quad (20)$$

$$a(w+W)_t = (w-W)(W-w)_{xx} + 2(W_x^2 + w_x W_x + w_x^2). \quad (21)$$

As far as we know, this one–singular manifold method always succeeds for PDEs admitting exactly one family of movable singularities. It usually fails for PDEs with more than one family, as summarized in Table 1. Its success for Sawada-Kotera, which has two families, has no explanation yet.

Table 1: Ability of the one–singular manifold method to find the DT, the Lax pair and/or the BT, according to the number of families of the PDE. "Opposite" is short for "families with opposite leading terms $\pm u_0 \chi^p$". For sine-Gordon, what is found is not exactly the DT, but the second solution U (we thank W. Schief for this precision).

PDE	Families	DT	Lax pair	BT
AKNS system	4	no	yes	no
Sine-Gordon	2 opposite	yes	no	no
MKdV	2 opposite	no	no	no
KdV	1	yes	yes	yes
Sawada-Kotera	2 non-opposite	yes	yes	yes [exception]
Kaup-Kupershmidt	2 non-opposite	no	no	no
Tzitzéica [21]	2 non-opposite	no	no	no

4 First attempt: the two–singular manifold method

While the one–singular manifold method extrapolates (5), the two–singular manifold method [16] extrapolates (6). The two basic assumptions (with some minor variations) are: the existence of a DT (8) with $\mathcal{D}_1 = -\mathcal{D}_2$ like in (6)

$$\text{DT}:\ u = \mathcal{D}(\operatorname{Log}\psi_1 - \operatorname{Log}\psi_2) + U \quad (22)$$

and the existence of a Riccati system satisfied by $\psi_1/\psi_2 = Y$. This results in a truncation $(p : -p)$ instead of $(p : 0)$

$$u = u_T = \sum_{j=0}^{-2p} u_j Y^{j+p},\ E_T = E(u_T) = \sum_{j=0}^{-2q} E_j Y^{j+q}. \quad (23)$$

As far as we know, it succeeds [16,5] for all PDEs admitting exactly two opposite families: sine-Gordon, MKdV, Broer-Kaup, but fails for AKNS.

5 Second attempt: Weiss plus homography

This method uses the freedom in the choice of the expansion variable.

Given a family $\varphi - \varphi_0 = 0$ (φ is a function, φ_0 an arbitrary constant), the expansion variable, which must vanish as $\varphi - \varphi_0$, is chosen as [4]

$$\chi = \left(\frac{\varphi_x}{\varphi - \varphi_0} - \frac{\varphi_{xx}}{2\varphi_x}\right)^{-1} = \frac{\psi}{\psi_x},\ \psi = (\varphi - \varphi_0)\varphi_x^{-\frac{1}{2}}. \tag{24}$$

Reciprocally, the most general expansion variable with the same property of generating homographically invariant coefficients (u_j, E_j) is [14]

$$Y = (A\chi^{-1} + B)^{-1}, \tag{25}$$

where A and B are two arbitrary homographically invariant functions.

Truncations $(p : 0)$ in χ and Y are equivalent (polynomials in χ^{-1} are polynomials in Y^{-1}), but a truncation $(p : -p)$ in Y may have solutions $B \neq 0$, i.e. more solutions than a truncation $(p : -p)$ in χ, see Ref. [19] for details. For instance, the Kaup-Kupershmidt equation admits a truncation $(-2 : 2)$ with $A = 1, B \neq 0$ and an arbitrary parameter λ [6], but this does not seem to provide the BT. For the AKNS system, this method also fails.

6 Present method: Weiss plus involutions

In case the one–singular manifold method fails to provide a BT but only provides some partial result $T(\chi, u, \lambda)$ for the truncation, one then considers all transformations on u conserving the equation $E(u) = 0$ in order to uncover a second solution U, see Table 2.

Table 2: Transformations of the dependent variable(s) conserving the equation(s), for the four PDEs of the AKNS group (complex conjugation, phase shift, parity).

PDE	Transformation(s)
AKNS system	$(u, v, i) \to (v, u, -i);\ \forall k:\ (u, v) \to (ku, v/k)$
Sine-Gordon	$u \to -u$
MKdV	$u \to -u$
KdV	none

For the AKNS system (3), the one–family truncation

$$u = u_0\chi^{-1} + u_1,\ v = v_0\chi^{-1} + v_1 \tag{26}$$

which has the general solution [23,16] (λ arbitrary complex constant)

$$
\begin{aligned}
u &= a(\chi^{-1} - f_x/(2f) - i\lambda)f && (27)\\
v &= a(\chi^{-1} + f_x/(2f) + i\lambda)/f && (28)\\
f_x/f &= -2i\lambda - (u/a)f^{-1} + (v/a)f && (29)\\
ip_r^{-1} f_t/f &= 2uv/a^2 + 4\lambda^2 + (u_x - 2i\lambda u)/(af) + (v_x + 2i\lambda v)f/a && (30)\\
(f_{xt} - f_{tx})/f &= (f^{-1}E^{(1)} + fE^{(2)})/a && (31)
\end{aligned}
$$

fails to introduce a second solution (U, V), see details in appendix C of Ref. [16]. This is done by applying the two point transformations of Table 2 to the above truncation T_1 (27)–(30):

$$
\left.
\begin{array}{lllllll}
T_1 : & \chi_1 & u & v & i & f & \lambda \quad \text{(identity)}\\
T_2 : & \chi_2 & v & u & -i & g & \mu \quad \text{(conjugation)}\\
T_3 : & \chi_3 & kU & k^{-1}V & i & f & \lambda' \quad \text{(phase shift)}\\
T_4 : & \chi_4 & k^{-1}V & kU & -i & g & \mu' \quad \text{(both)}
\end{array}
\right\} \qquad (32)
$$

These transformations act on (u, v, f, λ) like in Chen [3], not on (x, t) like in Yang and Schmid [25]. This is equivalent to successively process the four families of the AKNS system by the one–singular manifold method. In order that (u, v) and $(kU, V/k)$ be distinct, one must have $\lambda' = \mu, \mu' = \lambda$.

The four sets (27)–(28) define a system of eight equations in the eight unknowns $(\chi_1^{-1}, \chi_2^{-1}, \chi_3^{-1}, \chi_4^{-1}, u, v, kU, V/k)$. This system is linear with determinant $fg - 1/(fg)$ and it provides the DT straightforwardly (with the nonrestrictive choice $k = -1$):

$$
\begin{aligned}
u - U &= 2a[\partial_x \operatorname{Log}(g - 1/f) - i(\lambda + \mu)]/(g + 1/f) && (33)\\
v - V &= 2a[\partial_x \operatorname{Log}(f - 1/g) + i(\lambda + \mu)]/(f + 1/g) && (34)\\
u + U &= 2ia(\lambda - \mu)/(g - 1/f),\ v + V = 2ia(\lambda - \mu)/(f - 1/g) && (35)
\end{aligned}
$$

(to stick to our definition, the DT is made of two equations, either (33)–(34) or (35)). The nonconstant factor of the logarithmic derivatives is similar to that of (P3), (P5), (P6), see eq. (7). See Section 1 about $\lambda + \mu$.

The Lax pair in its Riccati form is made of the four equations resulting from the action of T_3 and T_4 on (29)–(30). The BT is made of the four equations resulting from the elimination of the two pseudopotentials (f, g) between the six equations defining the DT and the Lax pair, and these are precisely (4). This elimination is quite easy since equations (35) are algebraic in (f, g):

$$
f = ia(\lambda - \mu + R)/(v + V),\ g = ia(\lambda - \mu + R)/(u + U). \qquad (36)
$$

Remark. The "modified system" [3] of two equations for (f, g), obtained by eliminating (u, v, U, V) between (33)–(35) and the PDE, is invariant under exchange of (λ, μ). The elimination of g between this system provides the Broer-Kaup equation for $w = -i \operatorname{Log} f$, a result immediately obtainable by the Weiss truncation [16]

$$p_r^{-1} w_{tt} + 4 w_x w_{xt} + 2 w_t w_{xx} + p_r (6 w_x^2 w_{xx} + w_{xxxx}) = 0. \qquad (37)$$

7 Reductions of the DT of AKNS system

The "$x-$part" of the AKNS spectral problem admits the three reductions $v = \bar{u}, v = \pm u, v = 1$, and the DT, obtained only from the $x-$part, must admit them. This is indeed the case: equations (33)–(35) admit the two reductions $(v, V, g, \mu) = (\bar{u}, \bar{U}, \bar{f}, \bar{\lambda}), (\varepsilon u, \varepsilon U, \varepsilon f, -\lambda), \varepsilon^2 = 1$, and one must add the case of vanishing of the determinant $g = \varepsilon / f$. Table 3 summarizes these reductions and the homographic link between f and the χ of the invariant analysis.

Table 3: Reductions of the Darboux transformation of the AKNS system.

PDE	v, g, μ	χ^{-1}	$u - U$	$u + U$
NLS	$\bar{u}, \bar{f}, \bar{\lambda}$		(33)	$4a(\operatorname{Im} \lambda)/(1/f - \bar{f})$
SG MKdV	$\varepsilon u, \varepsilon f, -\lambda$ $e^2 = \varepsilon$	$\dfrac{\lambda}{Y} - \dfrac{eU}{4a}$ $Y = \dfrac{ef - 1}{ef + 1}$	$(4a/e) \dfrac{Y_x}{Y}$ $= \dfrac{2af_x}{\varepsilon f^2 - 1}$	$\dfrac{ia\lambda}{e} (\dfrac{1}{Y} - Y)$ $= 4ia\lambda \dfrac{f}{\varepsilon f^2 - 1}$
KdV	$1, \varepsilon / f, -\lambda$	$f - i\lambda$	$-2af_x$ $= -2a(\chi^{-1})_x$	$2a(f^2 - 2i\lambda f)$ $= 2a(\chi^{-2} + \lambda^2)$

8 Conclusion

Using Painlevé analysis *only*, one can find the Darboux transformation and the Lax pair, hence the Bäcklund transformation also for some PDEs which escape Weiss one–singular manifold method: one applies, first, Weiss method, secondly the discrete point symmetries. As compared to our previous "two-singular manifold method", the present method is simpler and, more important, it now succeeds for the ANKS system (NLS).

The next main difficulty to overcome is to handle PDEs with two non-opposite families, such as Kaup-Kupershmidt equation and Tzitzéica equation.

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