On a capillarity model and the Davey-Stewartson I system: Quasi-doubly periodic wave patterns

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Summary. – A phase field capillarity model due to Antanovskii is investigated under irrotational conditions. For certain two-parameter specific free energy functions and appropriate external driving conditions, the 2+1-dimensional nonlinear model is reduced to an integrable Davey-Stewartson system. Novel quasi-doubly periodic wave patterns are thereby generated. In the long wave limit, exponentially localized pulses are retrieved.

1. – Introduction

The classical theory of capillarity as originated by Gibbs [1] wherein two adjoining phases are assumed to be separated by a smooth geometric surface of zero thickness [2, 3] avoids the complex problem of the structure of the liquid-vapour separation region. Moreover, this zero thickness surface assumption does not allow the description of a wide range of physical situations involving capillarity including *inter alia*, the coalescence of touching drops, [4-6] the growth or dissolution of bubbles [7], the formation of pointed drops [8-11], the evolution of cusped interfaces [12-15], the rupture of thin films [16] and the dynamics of an advancing meniscus [17, 18].

Here, the concern is with a phase field model of capillarity developed *ab initio* by Antanovskii [19, 20] employing classical thermodynamic principles in combination with the balance of mass, momentum and energy. This capillarity model accounts for the structure of the liquid-vapour region in a systematic manner, and furthermore delivers an explicit expression for the capillary stress tensor in terms of the free energy of the liquid-vapour system. The classical expression for surface tension is retrieved from the model in the limit as the interfacial layer has vanishing thickness.

It was established by Antanovskii [19] that, if there is an external driving potential, then on neglecting viscous dissipation and heat conduction, the resultant model produces a self-consistent nonlinear system incorporating capillarity. Moreover, this system admits an important variational principle which was obtained earlier by Luke [21] and was subsequently discussed in detail by Seliger and Whitham [22]. It is with this externally driven, inviscid, isothermal capillarity model that the present note is concerned.

In the one dimensional unsteady case, it has been established by Antanovskii *et al* [23] that, in the absence of external driving mechanisms, for a certain class of model free energy functions, the capillarity model as set down [19] reduces to the classical nonlinear

Schrödinger (NLS) equation. Subsequently, for another class of model capillarity laws, the system was shown in [24] to reduce to a recently introduced variant of the NLS equation known as the resonant nonlinear Schrödinger equation (Pashaev and Lee [25]). The resonant NLS exhibits novel solitonic behavior and has recently been shown to arise in the analysis of the propagation of magneto-acoustic waves in cold plasma [26].

Here, it is established that, in the 2+1-dimensional case, for a two-parameter class of model free energy laws and appropriate external driving mechanisms, the capillarity model reduces to an integrable Davey-Stewartson system. A novel class of quasi-doubly periodic wave pattern solutions is presented for the latter and thereby for the capillarity model.

2. – The capillarity model: Reduction to the Davey-Stewartson I system

Here, we consider the following nonlinear system as set down by Antanovskii [19] for an inviscid, isothermal, externally driven fluid flow incorporating capillarity effects:

(1) $$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0\,,$$

(2) $$\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}=\nabla\left(\Pi-\frac{\delta}{\delta\rho}(\rho f)\right),$$

where ρ is the density, $\mathbf{u}$ is the velocity vector and Π is the potential for an external driving mechanism. In the above,

(3) $$\frac{\delta\psi}{\delta\rho}\equiv\frac{\partial\psi}{\partial\rho}-\nabla\cdot\left(\frac{\partial\psi}{\partial\alpha}\nabla\rho\right)$$

denotes the variational derivative, while $\alpha=\frac{1}{2}|\nabla\rho|^2$, and

(4) $$f=f(\rho,\alpha)\,,$$

designates the specific free energy. The quantity

(5) $$\zeta=\frac{\delta[\rho f]}{\delta\rho}$$

is the chemical potential of the liquid - vapour system.

In the one-dimensional case, for irrotational motion, the nonlinear system (1, 2) has been shown, for appropriate free energy relations (4) to reduce to the classical NLS equation [23] or to a variant known as the resonant NLS equation [25, 26].

In 2+1-dimensions, the integrable Davey-Stewartson system [27], with its origin in the work of Benney and Roskes [28], arises in the analysis of the propagation of capillary-gravity wave packets with streamwise, as well as spanwise modulations. The standard derivation employs weakly nonlinear, multi-scale perturbation theory. Here, by contrast, the Davey-Stewartson system is obtained as an exact 2+1-dimenstional reduction of the capillarity system (1, 2) without resort to perturbation theory. This reduction is established for a two-parameter class of free energy functions and an appropriate external driving mechanism.

Thus, in the case of irrotational flow subject to a conservative external force $\boldsymbol{\varepsilon}$, there exist potentials Φ and Π such that

(6) $$\mathbf{u} = \nabla\Phi\,, \quad \boldsymbol{\varepsilon} = \nabla\Pi$$

and the governing system (1, 2) reduces to

(7) $$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\nabla\Phi) = 0\,,$$

(8) $$\frac{\partial\Phi}{\partial t} + \frac{1}{2}\left|\nabla\Phi\right|^2 + \frac{\delta}{\delta\rho}(\rho f) - \Pi = B(t)\,.$$

The arbitrary function $B(t)$ in the Bernoulli integral (8) may be absorbed into the potential Φ and accordingly may be put zero without loss of generality.

If we set

(9) $$A = \rho^{1/2}\exp\left(\frac{i\Phi}{2}\right)\,,$$

then

(10) $$\frac{\partial A}{\partial t}=\frac{1}{2}\left[\rho^{-1/2}\frac{\partial\rho}{\partial t}+i\rho^{1/2}\frac{\partial\Phi}{\partial t}\right]\exp\left(\frac{i\Phi}{2}\right)\ ,$$

$$\nabla^2 A=\frac{1}{2}\left[\rho^{-1/2}\nabla^2\rho-\frac{1}{2}\rho^{-3/2}|\nabla\rho|^2+i\rho^{1/2}\nabla^2\Phi+i\rho^{-1/2}\nabla\rho\cdot\nabla\Phi-\frac{1}{2}\rho^{1/2}|\nabla\Phi|^2\right]\exp\left(\frac{i\Phi}{2}\right).$$

Hence, on using the continuity equation (7) and the Bernoulli integral (8) we obtain

(11) $$i\frac{\partial A}{\partial t}+\nabla^2 A=\frac{1}{2}\left[\frac{\delta}{\delta\rho}(\rho f)+\frac{\nabla^2\rho}{\rho}-\frac{1}{2}\frac{|\nabla\rho|^2}{\rho^2}-\Pi\right]A.$$

If attention is restricted to free energy functions of the type

(12) $$f=\frac{1}{2\rho^2}|\nabla\rho|^2+H(\rho)\ ,$$

where $H(\rho)$ is arbitrary, then (11) reduces to

(13) $$i\frac{\partial A}{\partial t}+\nabla^2 A+\left[J(|A|)+\frac{\Pi}{2}\right]A=0\,,$$

where

(14) $$J(|A|)=-\frac{1}{2}[\rho H(\rho)]'\quad ,\quad (|A|=\sqrt{\rho}\,)\,.$$

In this case, the capillary pressure adopts the form

(15) $$p\equiv\rho^2\frac{\delta f}{\delta\rho}=\rho^2 H'(\rho)-\rho\nabla^2(\log\rho)\,.$$

In particular, if $J(|A|)=v|A|^2$, so that

(16) $$H(\rho)=-v\rho+\frac{\mu}{\rho}\ ,$$

where μ and v are arbitrary parameters, then the evolution equation (13) becomes

(17) $$i\frac{\partial A}{\partial t}+\nabla^2 A+v|A|^2 A=-\frac{\Pi}{2}A\ .$$

In one spatial dimension and in the absence of external force (Π = 0), this reduces to the classical nonlinear Schrödinger equation. In 2+1-dimensions, if the external potential Π depending on two spatial variables ξ and η is constrained via

(18) $$\frac{\partial^2 \Pi}{\partial \xi^2} - \frac{\partial^2 \Pi}{\partial \eta^2} = -4v \frac{\partial^2}{\partial \xi^2}(AA^*),$$

then (17), (18) constitute the celebrated integrable Davey-Stewartson I (DSI) system. The model free energy functions for which this exact reduction applies adopt the two-parameter form

(19) $$f = \frac{1}{2}(\nabla \log \rho)^2 - v\rho + \frac{\mu}{\rho}.$$

The adoption of model constitutive laws for which governing equilibrium or dynamical equations reduce to analytically tractable form is well-established in nonlinear continuum mechanics [29]. Thus, the well-known Kaŕmán-Tsien pressure-density relation approximation and its variants in gasdynamics and the multi-parameter Cekirge-Varley stress-strain model laws in elastodynamics may be cited is this regard [30-33]. Therein, available parameters in the model laws are used to approximate real continuum behaviour over appropriate régimes. The two-parameter capillarity free energy model laws (19) for which the reduction to the integrable Davey-Stewartson I system holds are proposed with the same objective is mind. Additionally, the availability of exact analytic solutions of the nonlinear capillary system (1)-(2) with these constraints provides important necessary checks on numerical procedures. The generation of such exact solutions via the DSI integrable connection is the subject of the concluding section.

3. – Novel periodic and localized patterns

The generation of new kinds of exact solutions of the Davey-Stewartson I system has been the subject of recent research in [34, 35]. Here, we present novel periodic and localized wave patterns of the Davey-Stewartson I system.

If new coordinates X, Y are introduced via

(20) $$\xi = X + Y, \quad \eta = X - Y,$$

then the 2+1-dimensional Davey-Stewartson system (17)-(18) becomes

(21) $$i\frac{\partial A}{\partial t}+\frac{1}{2}\left(\frac{\partial^2 A}{\partial X^2}+\frac{\partial^2 A}{\partial Y^2}\right)+vA^2A^*=QA,$$

$$2\frac{\partial^2 Q}{\partial X\partial Y}=v\left(\frac{\partial}{\partial X}+\frac{\partial}{\partial Y}\right)^2(AA^*),$$

where $Q = -\Pi/2$. In [35], coherent solutions of the DS I system have been constructed by a novel separation of variables ansatz. This procedure is used here to generate a new class of exact solutions of the system (21) which exhibit quasi-doubly periodic structure. Thus, it is readily verified that the system admits a class of solutions dependent on two arbitrary functions p, q of one variable with

(22) $$A=\sqrt{\frac{2}{v}}\left(\frac{\sqrt{p_x q_y}}{p+q}\right)\exp(i(r+s)),\qquad Q=p_0+q_0-\frac{p_{xx}+q_{yy}}{p+q}+\frac{(p_x+q_y)^2}{(p+q)^2},$$

where

$$p=p(x),\qquad q=q(y),$$

(23) $$p_0=\frac{p_{xxx}}{4p_x}-\frac{p_{xx}^2}{8p_x^2}-\frac{\delta}{8}+\frac{c_1^2}{2}-\frac{\zeta^2}{2p_x^2},\qquad q_0=\frac{q_{yyy}}{4q_y}-\frac{q_{yy}^2}{8q_y^2}+\frac{\delta}{8}+\frac{c_2^2}{2}-\frac{\zeta^2}{2q_y^2}$$

$$r_x=c_1+\frac{\zeta}{p_x},\qquad s_y=c_2-\frac{\zeta}{q_y}$$

$$x=X-c_1t,\qquad y=Y-c_2t,$$

and δ, ζ, c_i, $i = 1, 2$ are arbitrary constants.

In our earlier work [35], a class of doubly periodic solutions for the Davey – Stewartson system is obtained by choosing odd powers of the Jacobi elliptic functions dn for p and q. Here we obtain another class of solutions by employing even powers of dn and setting

$$p(x)=\frac{a_0}{2}+\int_0^x \mathrm{dn}^2(m\tau,k)\,\mathrm{d}\tau,\qquad q(y)=\frac{a_0}{2}+\int_0^y \mathrm{dn}^2(n\tau,k^*)\,\mathrm{d}\tau,$$

where k, k^*, are independent elliptic functions moduli, and m, n are arbitrary real numbers. The resulting solutions of the DSI system typically exhibit quasi-doubly periodic behaviour for both $|A|^2$ and Q (Figure 1). This is in contrast with the classes of solutions of the DS I system presented [34, 35] and which produce strictly doubly-periodic wave patterns. It is noted that the quasi-doubly periodic waves obtained here propagate with constant speeds c_1, c_2 in the x, y directions respectively. It is remarked that, in the long wave limit with both k, k^* approaching unity, the preceding allows the construction of a novel, smooth localized solution of the DS I system with

$$p = \frac{a_0}{2} + \frac{\tanh mx}{m}, \quad q = \frac{a_0}{2} + \frac{\tanh ny}{n}, \quad \zeta = \delta = 0, \quad r = c_1 x, \quad s = c_2 y,$$

so that

$$A = \sqrt{\frac{2}{v}} \frac{\operatorname{sech} mx \operatorname{sech} ny \exp[i(r+s)]}{a_0 + (\tanh mx)/m + (\tanh ny)/n}.$$

The constraint $a_0 > |1/m| + |1/n|$ is required to avoid a singularity. Interestingly, this exact solution, while exhibiting exponential decay in both spatial directions, is distinct from the celebrated 'dromion' solution of the DS I system (Boiti *et al* [36], Fokas and Santini [37]).

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Figure Caption

(1) Quasi-doubly periodic patterns for the density or intensity $|A|^2$ of equation (22) plotted versus x and y, with $v = 2$, $a_0 = 20$, $m = 1$, $n = 1$, with moduli or nomes of the elliptic / theta functions in the x, y directions being 0.5 and 0.6 respectively.

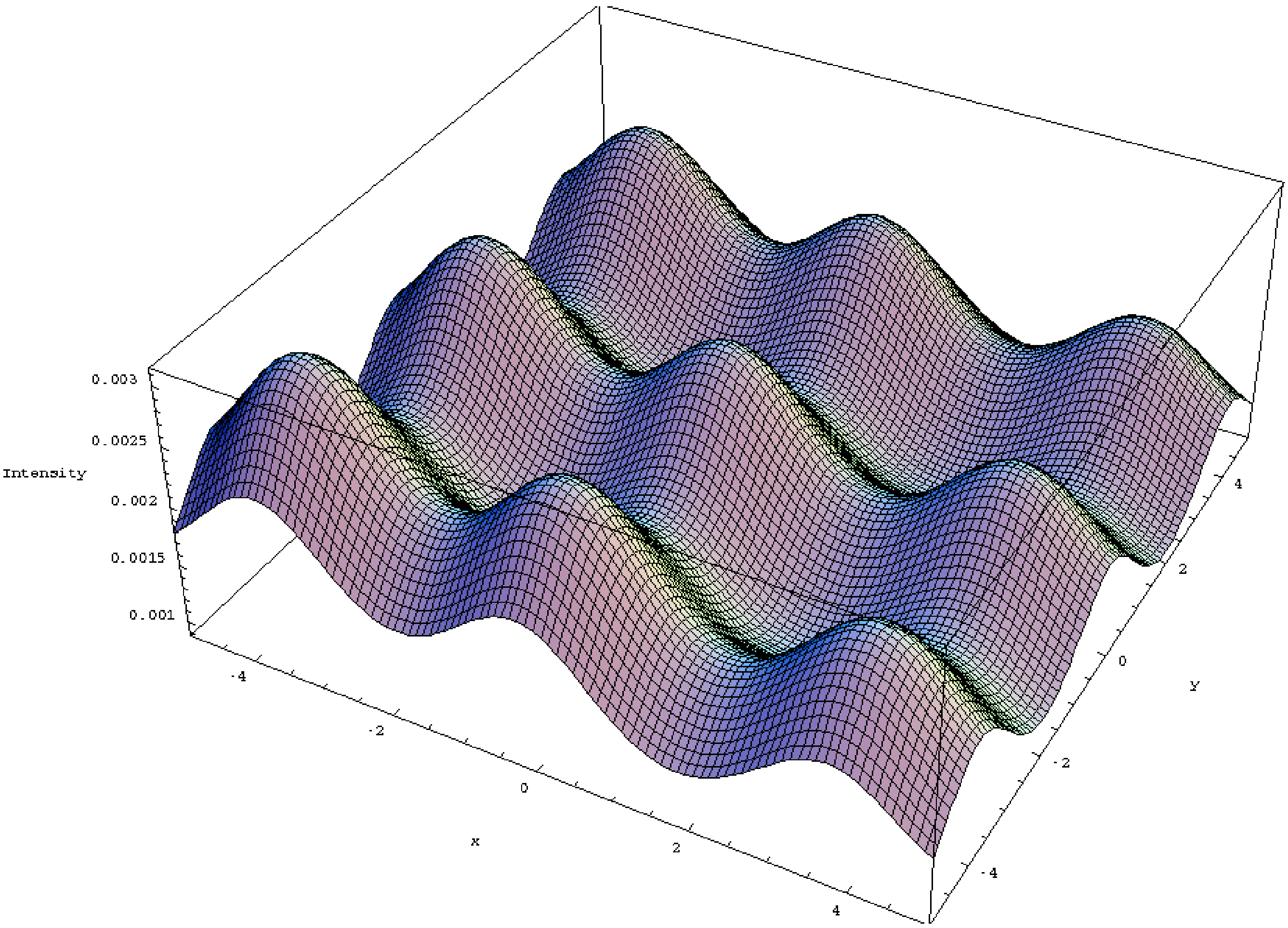

Figure 1