INTEGRATION OF PARTIALLY INTEGRABLE EQUATIONS

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Most evolution equations are partially integrable and, in order to explicitly integrate all possible cases, there exist several methods of complex analysis, but none is optimal. The theory of Nevanlinna and Wiman-Valiron on the growth of the meromorphic solutions gives predictions and bounds, but it is not constructive and restricted to meromorphic solutions. The Painlevé approach via the *a priori* singularities of the solutions gives no bounds but it is often (not always) constructive. It seems that an adequate combination of the two methods could yield much more output in terms of explicit (i.e. closed form) analytic solutions. We review this question, mainly taking as an example the chaotic equation of Kuramoto and Sivashinsky $\nu u''' + bu'' + \mu u' + u^2/2 + A = 0, \nu \neq 0$, with (ν, b, μ, A) constants.

1. Introduction

Phenomena in continuous media are often governed by a partial differential equation (PDE), e.g. in one space variable x and one time variable t

$$\mathbf{E}\left(\left\{\frac{\partial_{m+n}}{\partial x^m \partial t^n}\mathbf{u}\right\}\right) = 0, \tag{1}$$

in which $\mathbf{u}$ and $\mathbf{E}$ are multidimensional, the integers m, n take a finite set of values. Our interest is the nonintegrable or even chaotic case, for which the powerful tools of Lax pairs, inverse spectral transform, etc [1] are inapplicable. The derivation of analytic results must then use other methods. Let us quote a few examples.

(1) The one-dimensional cubic complex Ginzburg-Landau equation (CGL3)

$$iA_t + pA_{xx} + q|A|^2 A - i\gamma A = 0, \; pq\,\mathrm{Im}(p/q) \neq 0, \tag{2}$$

(and its complex conjugate, i.e. a total differential order four), in which p, q are complex constants and γ a real constant, a generic

equation which describes many physical phenomena, such as the propagation of a signal in an optical fiber [2], spatiotemporal intermittency in spatially extended dissipative systems [19,10,23]. For two coupled CGL3 equations, see analytic results in Ref. [6].

(2) The Kuramoto and Sivashinsky (KS) equation,

$$\varphi_t + \nu\varphi_{xxxx} + b\varphi_{xxx} + \mu\varphi_{xx} + \varphi\varphi_x = 0, \ \nu \neq 0, \tag{3}$$

in which ν, b, μ are real constants. This PDE is obeyed by the variable $\varphi = \arg A$ of the above field A of CGL3 under some limit [22,17], hence its name of phase turbulence equation.

(3) The quintic complex Ginzburg-Landau equation (CGL5),

$$iA_t + pA_{xx} + q|A|^2A + r|A|^4A - i\gamma A = 0, \ pr\,\mathrm{Im}(p/r) \neq 0, \tag{4}$$

in which p, q, r are complex constants and γ a real constant.

(4) The Swift-Hohenberg equation [24,18]

$$iA_t + bA_{xxxx} + pA_{xx} + q|A|^2A + r|A|^4A - i\gamma A = 0, \ br \neq 0, \tag{5}$$

in which b, p, q, r are complex constants and γ a real constant.

The autonomous nature of (1) (absence of any explicit dependence in x and t) allows the existence of *travelling waves* $u = U(\xi)$, solutions of the ordinary differential equation (ODE)

$$u(x,t) = U(\xi), \ \xi = x - ct, \ E(U^{(N)}, U^{(N-1)}, \ldots, U', U) = 0. \tag{6}$$

For the CGL3, KS, CGL5 and Swift-Hohenberg equations (with one exception, KS with $b^2 = 16\mu\nu$), all the solitary wave solutions $|A|^2 = f(\xi), \varphi = \Phi(\xi), \xi = x - ct$, which are known hitherto are polynomials in $\tanh k\xi$ (or $\cotanh, \tan, \cotan$, which are the same in the complex plane), and such solutions are easy to find by taking advantage of the singularity structure of the PDE (see, e.g., the summer school lecture notes [5]).

Hence the natural questions: (i) Can other solitary waves $u = f(x - ct)$ exist (in closed form)? (ii) If yes, please find them all, not just a few ones.

The present paper introduces to the methods in principle able to answer both questions. They will mainly be exemplified with the KS equation (3).

The paper is organized as follows. In section 2, we give a mathematical formulation of the problem. In section 3, we prove the inexistence of an analytic expression representing the *general solution*, and we compute the gap between the differential order N of the ODE (6) and the maximal number of integration constants in a singlevalued solution. In section 4, we give hints (not proofs) that some analytic result still has to be found. In

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section 5, we review the consequences of the assumption of singlevaluedness for a solution of the ODE (6), and present an algorithm to implement them. In section 6, we present the consequences of the assumption of meromorphy for a solution of (6). The last section 7 states the open problems.

2. Mathematical formulation of the problem

The successive steps of the announced program are

(1) To perform the traveling wave reduction from the PDE to an ODE. The KS PDE (3) depends on three *fixed* constants (ν, b, μ) (fixed means: which occur in the definition of the equation), the reduction

$$\varphi(x,t) = c + u(\xi), \ \xi = x - ct, \tag{7}$$

introduces in the ODE one more fixed constant A (the second constant c cancels out because of the Galilean invariance)

$$\nu u''' + bu'' + \mu u' + \frac{u^2}{2} + A = 0, \ \nu \neq 0, \tag{8}$$

and the general solution of (8), *if it exists*, depends on the four fixed constants (ν, b, μ, A) and three *movable* constants (movable means: which depends on the initial data), which are the origin ξ_0 of ξ and two other constants c_1, c_2.

(2) To count the number of constants which survive in the general solution of (8) when one requires singlevaluedness.

(3) To find this largest singlevalued particular solution in closed form. Indeed, its representation as a series can be misleading, as shown by classical authors like Poincaré and Painlevé.

3. Local separation of singlevaluedness and multivaluedness

Because the ODE (6) is nonintegrable, the number of integration constants present in any closed form solution is strictly smaller than the differential order of the ODE. This difference, an indicator of the amount of integrability of the ODE, can be precisely computed from a local analysis.

Two local representations of the general solution of (6) exist. The first one, also the most well known, is useless for our purpose. This is the famous Taylor series near a regular point, whose existence, unicity, convergence, etc is stated by the existence theorem of Cauchy. The reason why it is useless is its inability to make a distinction between chaotic ODEs such as (8) and integrable ODEs such as $u''' - 12uu' - 1 = 0$.

The second one, less known than the Taylor series of Cauchy, is a Laurent series (or more generally psi-series and/or Puiseux series) near a movable singularity x_0. This one does provide the expected information. The technique to compute it is just the Painlevé test (see Ref. [4] for the basic vocabulary of this technique). Let us present it on the KS example (8).

Looking for a local algebraic behaviour near a movable singularity x_0

$$u \sim_{x \to x_0} u_0 \chi^p, \ u_0 \neq 0, \ \chi = x - x_0, \tag{9}$$

one first balances the highest derivative and the nonlinearity,

$$p - 3 = 2p, \ p(p-1)(p-2)\nu u_0 + \frac{u_0^2}{2} = 0, \tag{10}$$

a system easily solved as

$$p = -3, \ u_0 = 120\nu. \tag{11}$$

The resulting convergent Laurent series,

$$u^{(0)} = \frac{120\nu}{\chi^3} - \frac{15b}{\chi^2} + \frac{15(16\mu\nu - b^2)}{4 \times 19\nu\chi} + \frac{13(4\mu\nu - b^2)b}{32 \times 19\nu^2} + O(\chi^1), \tag{12}$$

lacks two of the three arbitrary constants. They appear in perturbation [7],

$$u = u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \ldots, \tag{13}$$

in which ε is not in the ODE (8). The linearized equation around $u^{(0)}$

$$\left(\nu \frac{\mathrm{d}^3}{\mathrm{d}x^3} + b \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu \frac{\mathrm{d}}{\mathrm{d}x} + u^{(0)} \right) u^{(1)} = 0, \tag{14}$$

has then the Fuchsian type near x_0, with an indicial equation ($q = -6$ denotes the singularity degree of the lhs of (8))

$$\lim_{\chi \to 0} \chi^{-j-q} (\nu \partial_x^3 + u_0 \chi^p) \chi^{j+p} \tag{15}$$

$$= \nu(j-3)(j-4)(j-5) + 120\nu = \nu(j+1)(j^2 - 13j + 60). \tag{16}$$

The resulting local representation of the general solution,

$$\begin{aligned}
u(x_0, \varepsilon c_{-1}, \varepsilon c_+, \varepsilon c_-) = 120\nu\chi^{-3}\{&\text{Regular}(\chi) \\
&+\varepsilon[c_{-1}\chi^{-1}\text{Regular}(\chi) \\
&+ \ c_+\chi^{(13+i\sqrt{71})/2}\text{Regular}(\chi) \\
&+ \ c_-\chi^{(13-i\sqrt{71})/2}\text{Regular}(\chi)] + \mathcal{O}(\varepsilon^2)\},
\end{aligned} \tag{17}$$

in which "Regular" denotes converging series, depends on 4 arbitrary constants $(x_0, \varepsilon c_{-1}, \varepsilon c_+, \varepsilon c_-)$ but, as shown by Poincaré, the contribution of

εc_{-1} is the derivative of (12) with respect to x_0, so c_{-1} can be set to zero. The dense movable branching due to the irrational indices reflects [25] the chaos, and to remove it one has to require $\varepsilon c_+ = \varepsilon c_- = 0$, i.e. $\varepsilon = 0$, making the analytic part of (17) to depend on the single arbitrary constant x_0.

The ODE (8) admits other Laurent series in the variable $(u - \sqrt{-2A})^{-1}$, but they provide no additional information.

The question is then to turn this local information into a global one, i.e. to find the closed form singlevalued expression depending on the maximal number (here one) of movable constants.

We will call *unreachable* any constant of integration which cannot participate to any closed form solution. The KS ODE (8) has two unreachable integration constants, the third one x_0 being irrelevant since it reflects the invariance of (8) under a translation of x.

We will also call *general analytic solution* the closed form solution which depends on the maximal possible number of reachable integration constants, and our goal is precisely to exhibit a closed form expression for this general analytic solution, whose local representation is a Laurent series like (12).

The above notions (irrelevant, unreachable) belong to an equation, not to a solution. Let us introduce another integer number, attached to a solution, allowing one to measure its distance to the general analytic solution.

The *distance* of a closed form solution to the general analytic solution is defined as the number of constraints between the fixed constants and the reachable relevant constants.

For the ODE (8), the fixed constants are ν, b, μ, A, the movable constant x_0 is irrelevant, the movable constants $c_1 = \varepsilon c_+, c_2 = \varepsilon c_-$ are unreachable, so the distance d is the number of constraints among the fixed constants.

The closed form singlevalued solutions known to date are

(1) one elliptic solution (distance $d = 1$) [9,15]

$$b^2 = 16\mu\nu : \ u = -60\nu\wp' - 15b\wp - \frac{b\mu}{4\nu}, \ g_2 = \frac{\mu^2}{12\nu^2}, \ g_3 = \frac{13\mu^3 + \nu A}{1080\nu^3}, (18)$$

in which $\wp$ is the elliptic function of Weierstrass,

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3, \qquad (19)$$

(2) six trigonometric solutions ($d = 2$) [16,13], rational in $e^{k\xi}$,

$$u = 120\nu\tau^3 - 15b\tau^2 + \left(\frac{60}{19}\mu - 30\nu k^2 - \frac{15b^2}{4 \times 19\nu}\right)\tau$$

$$+ \frac{5}{2}bk^2 - \frac{13b^3}{32 \times 19\nu^2} + \frac{7\mu b}{4 \times 19\nu}, \ \tau = \frac{k}{2}\tanh\frac{k}{2}(\xi - \xi_0), (20)$$

the allowed values being listed in Table 1,

(3) one rational solution ($d = 3$),

$$b = 0, \ \mu = 0, \ A = 0 \ : \ u = 120\nu(\xi - \xi_0)^{-3}, \qquad (21)$$

which is a limit of all the previous solutions.

$b^2/(\mu\nu)$	$\nu A/\mu^3$	$\nu k^2/\mu$
0	$-4950/19^3, \ 450/19^3$	$11/19, \ -1/19$
$144/47$	$-1800/47^3$	$1/47$
$256/73$	$-4050/73^3$	$1/73$
16	$-18, \ -8$	$1, \ -1$

All those solutions admit the representation

$$u = \mathcal{D} \operatorname{Log} \psi + \text{constant}, \ \mathcal{D} = 60\nu\frac{\mathrm{d}^3}{\mathrm{d}\xi^3} + 15b\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \frac{15(16\mu\nu - b^2)}{76\nu}\frac{\mathrm{d}}{\mathrm{d}\xi}, \ (22)$$

in which ψ is an *entire* function. This linear operator $\mathcal{D}$, which captures the singularity structure, is called the *singular part operator*.

The Laurent series (12) yields another information [12]. If its sum is elliptic, the sum of the residues of the poles inside a period parallelogram must vanish. Since the only poles of (8) are one triple pole, a necessary condition [12] for the sum to be elliptic is to cancel the residue of (12), i.e. $b^2 = 16\mu\nu$. For this equation, the condition is also sufficient, see (18).

4. Experimental and numerical evidence of missing solutions

Experiments or computer simulations display regular patterns in the (x, t) plane (see [23]), some patterns being described by an analytic expression. For the other patterns, the guess is that there should exist matching analytic expressions. For the equation (3), one has observed a homoclinic wave [26] $\varphi = f(\xi), \xi = x - ct$, while all known solutions are heteroclinic.

The Laurent series (12) only provides a local knowledge of the general analytic solution. Rather than obtaining a global knowledge of the solution, which is the ultimate goal, it is easier to look at its singularities, by computing the *Padé approximants* [3] of the Laurent series (12). Padé approximants are a powerful tool to study the singularities of the sum of a given Taylor series, and more generally to perform the summation of divergent series.

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Given the first N terms of a Taylor series near $x = 0$,

$$S_N = \sum_{j=0}^{N} c_j x^j, \tag{23}$$

the Padé approximant $[L, M]$ of the series is the unique rational function

$$[L, M] = \frac{\sum_{l=0}^{L} a_l x^l}{\sum_{m=0}^{M} b_m x^m}, \ b_0 = 1, \tag{24}$$

obeying the condition

$$S_N - [L, M] = \mathcal{O}(x^{N+1}), \qquad L + M = N. \tag{25}$$

The extension to Laurent series presents no difficulty. In particular, for L and M large enough, Padé approximants are exact on rational functions .

The advantage of $[L, M]$ over S_N (which has no poles) is to display the *global structure* of singularities of the series.

From a thorough investigation [27] of the singularities of the sum of the Laurent series (12) one concludes (this is not a proof): for generic values of (ν, b, μ, A), no multivaluedness is detected, no cuts are detected, and the singularities look arranged in a nearly doubly periodic pattern, the elementary cell being made of one triple pole and three simple zeroes.

5. Consequences of singlevaluedness (Painlevé)

5.1. *Classical results on first order autonomous equations*

The failure to detect any multivaluedness in the unknown general analytic solution by no means implies the singlevaluedness of this general analytic solution, because the Painlevé test only generates necessary conditions, and the Padé approximants are a numerical investigation. It is however worthwhile to examine in detail the consequences of an assumed singlevaluedness.

Given the N-th order autonomous algebraic ODE (6), any solution is

$$u = f(\xi - \xi_0), \tag{26}$$

in which ξ_0 is movable. Provided the elimination of ξ_0 between the equation (26) and its derivative is possible, one obtains the first order nonlinear ODE

$$F(u, u') = 0, \tag{27}$$

in which F is as unknown as f.

However, $f(\xi - \xi_0)$ is now the *general solution* of (27), and there exist classical results on first order autonomous ODEs which are in addition

algebraic. Let us therefore assume from now on that the dependence of f on ξ_0 is algebraic (this is a sufficient condition for F to be algebraic).

Let us summarize. Given the N-th order ODE (6) and its particular solution f Eq. (26), and assuming the dependence of f on ξ_0 to be algebraic, one is able to derive a first order ODE (27) which is algebraic.

Conversely, given an algebraic first order ODE $F = 0$ Eq. (27), is it possible to go back to f? This question has been answered positively by Briot and Bouquet, Fuchs, Poincaré and put in final form by Painlevé [21] .

Theorem 5.1. *Given the algebraic first order ODE $F = 0$ Eq. (27), if its general solution is singlevalued, then*

(1) *Its general solution is an elliptic function, possibly degenerate, and its expression is known in closed form.*
(2) *The genus of the algebraic curve (27) is one or zero.*
(3) *There exist a positive integer m and $(m + 1)^2$ complex constants $a_{j,k}$, with $a_{0,m} \neq 0$, such that the polynomial F has the form*

$$F(u, u') \equiv \sum_{k=0}^{m} \sum_{j=0}^{2m-2k} a_{j,k} u^j u'^k = 0, \ a_{0,m} \neq 0. \qquad (28)$$

Then, assuming f singlevalued with an algebraic dependence on ξ_0,

(1) It is equivalent to search for the solution f or for F.
(2) The solution f can only be elliptic (i.e. rational in $\wp$ and $\wp'$), or a rational function of e^{ax} with a constant, or a rational function of x.

The explicit form (28) of F makes it much easier to look for F than f.

5.2. *Method to obtain the first order autonomous subequation*

The input data and assumptions are:

(1) a N-th order algebraic ODE (6), $N \geq 2$,
(2) a Laurent series representing its general analytic solution,
(3) a first order algebraic ODE sharing its general solution with (6).

Then, by the classical results of section 5.1, F has the form (28), and there exists an algorithm [20] yielding the solution f in the canonical form

$$u = R(\wp', \wp) = R_1(\wp) + \wp' R_2(\wp), \qquad (29)$$

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in which R_1, R_2 are two rational functions, with the possible degeneracies

$$R(\wp', \wp) \longrightarrow R(e^{k\xi}) \longrightarrow R(\xi), \tag{30}$$

in which R denotes rational functions. This algorithm is [20] :

(1) Compute finitely many terms of the Laurent series,

$$u = \chi^p \left(\sum_{j=0}^{J} u_j \chi^j + \mathcal{O}(\chi^{J+1}) \right), \ \chi = \xi - \xi_0. \tag{31}$$

(2) Choose a positive integer m and define the first order ODE

$$F(u, u') \equiv \sum_{k=0}^{m} \sum_{j=0}^{[(m-k)(p-1)/p]} a_{j,k} u^j u'^k = 0, \ a_{0,m} \neq 0, \tag{32}$$

in which $[z]$ denotes the integer part function. The upper bound on j implements the condition $m(p-1) \leq jp + k(p-1)$, identically satisfied if $p = -1$, that no term can be more singular than u'^m.

(3) Require the Laurent series to satisfy the Briot and Bouquet ODE, i.e. require the identical vanishing of the Laurent series for the lhs $F(u.u')$ up to the order J

$$F \equiv \chi^{m(p-1)} \left(\sum_{j=0}^{J} F_j \chi^j + \mathcal{O}(\chi^{J+1}) \right), \ \forall j \ : \ F_j = 0. \tag{33}$$

If it has no solution for $a_{j,k}$, increase m and return to first step.

(4) For every solution, integrate the first order autonomous ODE (32).

The main step is to solve the set of equations (33), i.e. a linear, overdetermined system in the unknowns $a_{j,k}$. This is quite an easy task.

An upper bound on m will be established in section 6.

5.3. *Results of the method on the KS equation*

The Laurent series of (8) is (12). In the second step, the smallest integer m allowing a triple pole ($p = -3$) in (32) is $m = 3$. With the normalization $a_{0,3} = 1$, the subequation contains ten coefficients, which are first determined by the Cramer system of ten equations $F_j = 0, j = 0 : 6, 8, 9, 12$. The remaining overdetermined nonlinear system for (ν, b, μ, A) contains as

greatest common divisor (gcd) $b^2 - 16\mu\nu$, which defines a first solution

$$\frac{b^2}{\mu\nu} = 16, \ u_s = u + \frac{3b^3}{32\nu^2},$$

$$\left(u' + \frac{b}{2\nu}u_s\right)^2 \left(u' - \frac{b}{4\nu}u_s\right) + \frac{9}{40\nu}\left(u_s^2 + \frac{15b^6}{1024\nu^4} + \frac{10A}{3}\right)^2 = 0. \quad (34)$$

After division by this gcd, the remaining system for (ν, b, μ, A) admits four solutions (stopping the series at $J = 16$ is enough), namely the first three lines of Table 1, each solution defining a subequation,

$$b = 0,$$

$$\left(u' + \frac{180\mu^2}{19^2\nu}\right)^2 \left(u' - \frac{360\mu^2}{19^2\nu}\right) + \frac{9}{40\nu}\left(u^2 + \frac{30\mu}{19}u' - \frac{30^2\mu^3}{19^2\nu}\right)^2 = 0, \quad (35)$$

$$b = 0, \ u'^3 + \frac{9}{40\nu}\left(u^2 + \frac{30\mu}{19}u' + \frac{30^2\mu^3}{19^3\nu}\right)^2 = 0, \quad (36)$$

$$\frac{b^2}{\mu\nu} = \frac{144}{47}, \ u_s = u - \frac{5b^3}{144\nu^2}, \ \left(u' + \frac{b}{4\nu}u_s\right)^3 + \frac{9}{40\nu}u_s^4 = 0, \quad (37)$$

$$\frac{b^2}{\mu\nu} = \frac{256}{73}, \ u_s = u - \frac{45b^3}{2048\nu^2},$$

$$\left(u' + \frac{b}{8\nu}u_s\right)^2 \left(u' + \frac{b}{2\nu}u_s\right) + \frac{9}{40\nu}\left(u_s^2 + \frac{5b^3}{1024\nu^2}u_s + \frac{5b^2}{128\nu}u'\right)^2 = 0,$$

$$(38)$$

To integrate the subequations (34), (35)–(38), one first computes their genus [a], which is one for (34), and zero for (35)–(38). Therefore (34) has an elliptic general solution, listed above as (18). The general solution of the four others (35)–(38) is the third degree polynomial (20) in $\tanh k(\xi - \xi_0)/2$.

These four solutions, obtained for the minimal choice of the subequation degree m, constitute all the analytic results currently known on (8).

6. Consequences of meromorphy (Nevanlinna)

If the solution f is meromorphic, much can be said from the study of its growth at infinity (Nevanlinna theory). For the KS ODE, the meromorphy requires $c_+ = c_- = 0$ in (17), restricting the solution to the series (12).

By direct application of the Nevanlinna theory, one can prove the

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[a]For instance with the Maple command *genus* of the package *algcurves* [11], which implements an algorithm of Poincaré.

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Theorem 6.1. [8] *If a solution of (8) is meromorphic, then it is elliptic or degenerate of elliptic. Furthermore,*

(1) *Elliptic solutions only exist if $b^2 = 16\mu\nu$, and their order is three.*
(2) *Exponential solutions have the necessary form $P(\tanh k(\xi - \xi_0))$, with k constant and P a polynomial of degree three.*
(3) *The only rational solution is $u = 120\nu(\xi - \xi_0)^{-3}$, it exists for $b = \mu = A = 0$.*

Consequently, the value $m = 3$ is an upper bound to the algorithm of section 5.2, which has therefore found all the meromorphic solutions of (8).

7. Summary and open problems

Let us represent the solutions of (8) by the following inclusions,

$$\text{elliptic} \subset \text{meromorphic} \subset \text{singlevalued} \subset \text{multivalued}. \qquad (39)$$

One has seen the various implications

(1) (Singlevalued, algebraic dependence on x_0) $\Longrightarrow$ elliptic (thm 5.1),
(2) Meromorphic $\Longrightarrow$ elliptic ([8] using Nevanlinna theory),
(3) Elliptic $\Longrightarrow$ ($b^2 = 16\mu\nu$) (residue theorem [12]),
(4) (Elliptic or degenerate) $\Longrightarrow$ (order three) ([8] using Nevanlinna theory) $\Longrightarrow$ (all such solutions in closed form [20]).

The problem is open to find the general analytic solution in closed form for arbitrary (ν, b, μ, A), which would be the sum of the Laurent series (12). Padé approximants and Painlevé analysis find no multivaluedness nowhere.

Two and only two possibilities remain about this general analytic solution for generic values of (ν, b, μ, A),

(1) either it is multivalued, and strong efforts have then to be made to uncover this multivaluedness with both the Painlevé test and the Padé approximants. This event is unlikely;
(2) or it is singlevalued. In this case it cannot be elliptic, and the dependence on x_0 is necessarily transcendental.

Solving this open problem would solve *ipso facto* many similar problems for nonintegrable equations such as CGL3, CGL5 or Swift-Hohenberg.

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